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TECHNICAL NOTE 3009

VELOCITY POTENTIAL AND AIR FORCES  
ASSOCIATED WITH A TRIANGULAR WING IN SUPERSONIC FLOW,  
WITH SUBSONIC LEADING EDGES, AND DEFORMING HARMONICALLY  
ACCORDING TO A GENERAL QUADRATIC EQUATION

By Charles E. Watkins and Julian H. Berman

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## SUMMARY

The velocity potential for a triangular wing with subsonic leading edges experiencing harmonic deformations in supersonic flow is treated herein. The oscillations considered are such that the amplitude of distortion of the wing can be represented by a general quadratic expression for a surface. The velocity potential is obtained in the form of a power series in terms of the frequency of oscillation. Although only terms appropriate for expressing the potential to the third power of the frequency are presented, additional terms may be obtained if they are desired. The material constitutes an extension of work given in NACA Report 1099.

## INTRODUCTION

Designers of supersonic aircraft are leaning more and more toward the use of triangular plan forms for wings and control surfaces. In order to obtain information concerning the aeroelastic properties of such wings, knowledge is needed of the air forces that may act on them.

In flutter calculations for wings whose deformations can be calculated satisfactorily by simple-beam theory, the use of aerodynamic coefficients based on harmonic pitching and translation of representative sections of nondistorting, or rigid, wings has proved reasonably satisfactory. Simple-beam theory, however, does not as readily apply to triangular wings or to swept wings of small aspect ratio and for a structural or a flutter analysis of these wings recourse must be had to a more appropriate theory. The structural analysis would define the structural deformations or the natural modes of the wing, and the flutter analysis would involve the use of aerodynamic coefficients related to these structural deformations or modes.

In references 1 and 2 use is made of classical plate theory to develop a method for calculating stresses and deformations of thin, homogeneous, cantilever wings of arbitrary plan form. Results of this method when applied to a triangular plan form indicate that aerodynamic coefficients associated with sinusoidal distortions including deformations of the second degree in bending and camber might be useful in flutter analyses of triangular wings, particularly if a modal-analysis approach is used.

In references 3 and 4 aerodynamic forces and moments on rigid triangular wings harmonically oscillating in pitch and in vertical translation in supersonic flow are derived by a method of expanding the associated velocity potentials in terms of the frequency of oscillation. This method of expansion can also be used to derive the potentials associated with harmonically distorting wings, provided the form of distortion is known. This method, of course, applies to triangular wings with both supersonic and subsonic leading edges, but only the subsonic-leading-edge case, which is theoretically the more difficult one, is treated herein.

The main purpose of the present paper is, therefore, to obtain the expanded velocity potential for a triangular wing with subsonic leading edges undergoing general second-degree forms of harmonic distortion in both spanwise and chordwise directions and to present the first few terms of the expansion. Such a potential can be made to yield not only flutter coefficients for a distorting triangular wing but also certain steady-state and time-dependent stability derivatives.

For triangular wings with supersonic leading edges the treatment is a straightforward procedure, since the boundary-value problem for the velocity potential in this case can be satisfied, as shown in reference 5, by simple distributions of sources with local strength proportional to local downwash. Although this case is not directly dealt with herein, the "sonic case," or the case where the Mach lines from the vertex of the triangular wing coincide with the wing leading edges, which can be considered as belonging to either the case of subsonic or supersonic leading edges, is used as a check, in the limiting case, on the results presented herein.

It is of interest to note that the circular plan form in incompressible flow has been treated for the same forms of harmonic distortions considered herein (ref. 6). A useful byproduct of such treatments is that they furnish means by which various strip-theory approaches for obtaining aerodynamic forces and moments for a deforming wing, as commonly used in flutter analyses, may be evaluated.

## SYMBOLS

$A, B, C, D, E, F$	constants defined in equation (4)
$\left. \begin{matrix} A_1, B_1, C_1, D_1, E_1, F_1 \\ \bar{A}_1, \bar{B}_1, \bar{C}_1, \bar{D}_1, \bar{E}_1, \bar{F}_1 \end{matrix} \right\}$	constants depending on $\rho_0$ defined in appendix
$a, b, c, d, e, f$	coefficients used in equation (3) to define displacement of wing
$a_1, b_1, c_1, d_1, e_1, f_1$	constants depending on $\rho_0$ and $M$
$\left. \begin{matrix} a_1', b_1', c_1', d_1', e_1', f_1' \\ a_1'', b_1'', c_1'', d_1'', e_1'', f_1'' \end{matrix} \right\}$	constants depending on $\rho_0$
$a_{nm}$	functions of $\bar{\omega}$ , $x$ , and $M$
$2b$	root chord of wing
$c$	velocity of sound
$F', E'$	complete elliptic integrals of first and second kinds, respectively, with modulus $\sqrt{1 - \rho_0^2}$
$G_n, \bar{G}_n, \bar{\bar{G}}_n$	doublet distribution functions
$k$	reduced frequency, $b\omega/V$
$M$	free-stream Mach number, $V/c$
$\Delta p$	local pressure difference
$R$	distance, $\sqrt{(x - \xi)^2 - \beta^2(y - \eta)^2 - \beta^2 z^2}$
$r$	region of integration (see fig. 1)
$t$	time
$V$	free-stream velocity
$W(x, y, t)$	vertical velocity at surface of wing
$W_{n,m}^p, \bar{W}_{n,m}^p$	constants associated with doublet distribution functions depending on $\rho_0$ and $\beta$

$x, y, z$	rectangular coordinates attached to wing moving in negative x-direction
$Z_m(x, y, t)$	sinusoidal displacement of a point on wing
$\beta = \sqrt{M^2 - 1}$	
$\lambda$	tangent of half-apex angle of triangular wing
$\xi, \eta$	rectangular coordinates used to represent space location of doublets in xy-plane
$\rho$	density
$\rho_0 = \beta \lambda$	
$\phi$	disturbance-velocity potential
$\omega$	frequency of oscillations
$\bar{\omega} = M^2 \omega / V \beta^2$	

### ANALYSIS

#### The Boundary-Value Problem

The differential equation, referred to a rectangular coordinate system moving forward at uniform speed  $V$  in the negative x-direction, that must be satisfied by the velocity potential  $\phi$  is

$$\frac{1}{c^2} \left( \frac{\partial}{\partial t} + V \frac{\partial}{\partial x} \right)^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \quad (1)$$

The main governing boundary condition, that of tangential flow at the wing surface, is

$$\left( \frac{\partial \phi}{\partial z} \right)_{z=0} = W(x, y, t) = \left( V \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right) Z_m(x, y, t) \quad (2)$$

where  $W(x,y,t)$  and  $Z_m(x,y,t)$  represent, respectively, the vertical velocity (or downwash) and vertical displacement of any point  $(x,y,z)$  of the wing. Let the amplitude of vertical displacement of the wing be determined by a general second-degree equation in  $x$  and  $y$ ; then the displacement of any point of the harmonically oscillating wing may be expressed as

$$Z_m = e^{i\omega t}(ax^2 + by^2 + cxy + dx + ey + f) \quad (3)$$

where the coefficients  $a, b, c, \dots$  may be considered as complex quantities in order to permit the inclusion of phase differences between the different components of the displacement. By substituting the expression for  $Z_m$  of equation (3) into equation (2), an expression for the downwash is obtained which may be written as

$$\begin{aligned} \left(\frac{\partial \phi}{\partial z}\right)_{z=0} &= W(x,y,t) \\ &= e^{i\omega t}(Ax^2 + B\beta^2 y^2 + C\beta xy + Dx + E\beta y + F) \end{aligned} \quad (4)$$

The commonly used Mach number relation  $\beta \equiv \sqrt{M^2 - 1}$  has been introduced into the coefficients of the  $y$ -terms in equation (4) for convenience in the subsequent analysis. The coefficients  $A, B, C, \dots$  are related to the coefficients  $a, b, c, \dots$  in equation (3) in the following manner:

$$\begin{aligned} A &= i\omega a & C &= \frac{i\omega c}{\beta} & E &= \frac{Vc + i\omega e}{\beta} \\ B &= \frac{i\omega b}{\beta^2} & D &= 2aV + i\omega d & F &= Vd + i\omega f \end{aligned}$$

Of course, for some purposes it may be more logical to start with a prescribed downwash or vertical velocity rather than a prescribed deformation.

As is usual in dealing with linearized aerodynamic problems, the velocity is conveniently expressed as a sum of separate effects associated with the different terms of the downwash expression, equation (4), namely

$$\phi = \phi_A + \phi_B + \phi_C + \phi_D + \phi_E + \phi_F \quad (5)$$

With the potential expressed in this form attention may be directed to the derivation of these subsidiary potentials  $\phi_A$ ,  $\phi_B$ , and so forth, individually.

#### Derivation of Velocity Potential $\phi$

Method of solution.- The boundary-value problem for each of the subsidiary potentials in equation (5) can be satisfied by the method discussed in reference 4 of expanding the velocity potential to any given power  $\nu$  of the frequency parameter  $\bar{\omega} = M^2\omega/V\beta^2$ . The procedure for obtaining any one of these potentials is essentially the same as that for obtaining any other. It will therefore suffice to discuss the derivation of only one subsidiary potential, say  $\phi_A$ . The observations, pertaining to pulsating doublets, made in the following paragraph may be helpful in the subsequent outline and discussion of the derivation.

When the potential of a doublet that is pulsating with frequency  $\omega$  and moving at a uniform speed  $V$  is expanded in terms of  $\bar{\omega}$ , certain terms of the expansion are found to contain as a factor the so-called steady-state doublet potential  $z/R^3$  (hereinafter referred to as singular terms), whereas all other terms contain factors of the type  $(\bar{\omega}/M)^{2m-2}zR^{2m-5}$ ,  $m \geq 2$  (hereinafter referred to as nonsingular terms). Of these terms, only the singular ones contribute directly to the potential for the airfoil; nevertheless, both singular and nonsingular terms are, in general, required to satisfy the differential equation, equation (1), to a given power of  $\bar{\omega}$ . Furthermore, when these expanded potentials are employed in satisfying the boundary conditions, the nonsingular terms give rise to downwash that, as indicated by the factor  $(\bar{\omega}/M)^{2m-2}$ , is of higher degree in  $\bar{\omega}$  than the downwash arising from the singular terms, an observation that provided a key to the step-by-step procedure adopted in reference 4.

In this procedure the singular terms can be grouped and weighted so that they alone satisfy the specified downwash conditions. At the same time the downwash arising from the nonsingular terms is to be canceled to the required power of  $\bar{\omega}$  by the introduction of other doublet potentials in order to preserve the downwash conditions satisfied by the singular terms. It is through this canceling process that the nonsingular terms give rise, indirectly, to any contributions to the potential for the airfoil.

The distribution functions.- In accordance with the preceding discussion, the potential under discussion can conveniently be considered as a sum, namely

$$\phi_A = \phi_1 + \bar{\phi}_1 + \bar{\bar{\phi}}_1 + \dots \quad (6)$$

where  $\phi_1$  is so treated that the singular terms contained therein alone can be made to satisfy the boundary conditions and the terms  $\bar{\phi}_1, \bar{\bar{\phi}}_1, \dots$  are successively required to cancel the downwash arising from the non-singular terms of  $\phi_1$ . The term  $\phi_1$ , which may be considered as a summation of weighted doublets expanded to the  $v$ th power of  $\bar{\omega}$  and integrated over the appropriate portion  $r$  (see fig. 1) of the wing plan form, may be written as

$$\begin{aligned} \phi_1 &= \frac{Ae^{i\omega t}}{\pi} \lim_{z \rightarrow 0} \frac{\partial}{\partial z} \sum_{n=0}^v \left[ a_{n1} \iint_r G_n(\xi, \eta) \frac{\xi^{n+1}}{R} d\xi d\eta + \right. \\ &\quad \left. \sum_{m=2}^4 \frac{2v+3+(-1)^v}{4} a_{nm} \iint_r G_n(\xi, \eta) \xi^{n+1} R^{2m-3} d\xi d\eta \right] \\ &= Ae^{i\omega t} \sum_{n=0}^v a_{n1} x^n G_n(x, y) \end{aligned} \quad (7)$$

where

$$R = \sqrt{(x - \xi)^2 - \beta^2(y - \eta)^2 - \beta^2 z^2}$$

and

$$a_{n1} = \frac{(i\bar{\omega})^n}{n!} \left[ 1 - i\bar{\omega}x - \frac{\bar{\omega}^2 x^2}{2!} + \dots + \frac{(-i\bar{\omega}x)^{v-n}}{(v-n)!} \right] \quad (8)$$

$$a_{nm} = \frac{(-1)^{m-1}}{(2m-2)!} \left( \frac{\bar{\omega}}{M} \right)^{2m-2} \frac{(i\bar{\omega})^n}{n!} \left[ 1 - i\bar{\omega}x + \dots + \frac{(-i\bar{\omega}x)^{v+2-n-2m}}{(v+2-n-2m)!} \right] \quad (9)$$



The functions  $G_n(\xi, \eta)$  represent weighting or distribution functions that are to be determined so that the first set of integrals in the brackets in equation (7) (which, as may be noted, employ as kernels the singular terms of the expanded doublet potential) satisfy the prescribed downwash condition (see ref. 4 for a more detailed discussion), namely

$$\lim_{z \rightarrow 0} \left( \frac{\partial \phi_A}{\partial z} \right) = A e^{i\omega t} x^2 \quad (10)$$

Corresponding expressions for  $\bar{\phi}_1, \bar{\bar{\phi}}_1, \dots$  may be written as follows:

$$\begin{aligned} \bar{\phi}_1 &= \frac{A e^{i\omega t}}{\pi} \lim_{z \rightarrow 0} \frac{\partial}{\partial z} \left( \frac{\bar{\omega}^2}{2M^2} \right) \sum_{n=0}^{\nu-2} \left[ a_{n1} \iint_r \bar{G}_n(\xi, \eta) \frac{\xi^{n+3}}{R} d\xi d\eta + \right. \\ &\quad \left. \frac{2\nu-1+(-1)^\nu}{\sum_{m=2}^4} a_{nm} \iint_r \bar{G}_n(\xi, \eta) \xi^{n+3} R^{2m-3} d\xi d\eta \right] \\ &= A e^{i\omega t} \sum_{n=0}^{\nu-2} \frac{\bar{\omega}^2}{2M^2} a_{n1} x^{n+2} \bar{G}_n(x, y) \end{aligned} \quad (11)$$

$$\begin{aligned} \bar{\bar{\phi}}_1 &= \frac{A e^{i\omega t}}{\pi} \lim_{z \rightarrow 0} \frac{\partial}{\partial z} \left( \frac{\bar{\omega}^4}{4M^4} \right) \sum_{n=0}^{\nu-4} \left[ a_{n1} \iint_r \bar{\bar{G}}_n(\xi, \eta) \frac{\xi^{n+5}}{R} d\xi d\eta + \right. \\ &\quad \left. \frac{2\nu-5+(-1)^\nu}{\sum_{m=2}^4} a_{nm} \iint_r \bar{\bar{G}}_n(\xi, \eta) \xi^{n+5} R^{2m-3} d\xi d\eta \right] \\ &= A e^{i\omega t} \sum_{n=0}^{\nu-4} \frac{\bar{\omega}^4}{4M^4} a_{n1} x^{n+4} \bar{\bar{G}}_n(x, y) \end{aligned} \quad (12)$$

...

where the functions  $\bar{G}_n(\xi, \eta)$ ,  $\bar{\bar{G}}_n(\xi, \eta)$ , . . . are to be determined so that  $\bar{\phi}_1, \bar{\bar{\phi}}_1, \dots$  yield a downwash that will cancel the downwash arising from the second set of integrals in the equation for  $\phi_1$ , equation (7), which employ nonsingular terms as kernels.

Since the following identity holds (see eq. (13) of ref. 4),

$$\sum_{n=0}^{\nu} x^n a_{n1} \equiv 1$$

it follows that the integral equations which determine successively the distribution functions  $G_n, \bar{G}_n, \bar{\bar{G}}_n, \dots$  ( $n = 0, 1, 2, 3, \dots$ ) so that the condition of tangential flow for  $\phi_A$  is satisfied are as follows:

$$\frac{A}{\pi} \lim_{z \rightarrow 0} \frac{\partial^2}{\partial z^2} \iint_r G_n(\xi, \eta) \frac{\xi^{n+1}}{R} d\xi d\eta = Ax^2(x^n) = Ax^{n+2} \quad (13)$$

$$\lim_{z \rightarrow 0} \frac{\partial^2}{\partial z^2} \iint_r \bar{G}_n(\xi, \eta) \frac{\xi^{n+3}}{R} d\xi d\eta = \lim_{z \rightarrow 0} \frac{\partial^2}{\partial z^2} \iint_r G_n(\xi, \eta) \xi^{n+1} R d\xi d\eta \quad (14)$$

$$\lim_{z \rightarrow 0} \frac{\partial^2}{\partial z^2} \iint_r \bar{\bar{G}}_n(\xi, \eta) \frac{\xi^{n+5}}{R} d\xi d\eta = \lim_{z \rightarrow 0} \frac{\partial^2}{\partial z^2} \left[ -\frac{1}{6} \iint_r G_n(\xi, \eta) \xi^{n+1} R^3 d\xi d\eta + \iint_r \bar{G}_n(\xi, \eta) \xi^{n+3} R d\xi d\eta \right] \quad (15)$$

These integral equations may be solved by the method discussed in the appendix of reference 4 where, in view of integration difficulties, it is found necessary to consider not only the downwash that is to be satisfied but also the derivatives of the downwash. This method, though simple in principle, is rather lengthy to apply; consequently, only the results of the solutions will be given herein. Furthermore,

the potentials will hereinafter be restricted to terms up to the third power of  $\bar{\omega}$  (that is,  $\nu = 3$ ). To this power of  $\bar{\omega}$  the distribution functions  $\bar{G}_n$ , equation (15), are not required.

Expressions for  $G_n$  and  $\bar{G}_n$  are found to be as follows:

$$\left. \begin{aligned} G_0 &= \xi^2 \left[ A_0 + A_1 \frac{\beta^2 \eta^2}{\xi^2} \sqrt{\lambda^2 \xi^2 - \eta^2} \right] \\ G_1 &= \xi^2 \left[ A_2 + A_3 \frac{\beta^2 \eta^2}{\xi^2} \sqrt{\lambda^2 \xi^2 - \eta^2} \right] \\ G_2 &= \xi^2 \left[ A_4 + A_5 \frac{\beta^2 \eta^2}{\xi^2} + A_6 \frac{\beta^4 \eta^4}{\xi^4} \sqrt{\lambda^2 \xi^2 - \eta^2} \right] \\ G_3 &= \xi^2 \left[ A_7 + A_8 \frac{\beta^2 \eta^2}{\xi^2} + A_9 \frac{\beta^4 \eta^4}{\xi^4} \sqrt{\lambda^2 \xi^2 - \eta^2} \right] \\ \bar{G}_0 &= \xi^2 \left[ \bar{A}_0 + \bar{A}_1 \frac{\beta^2 \eta^2}{\xi^2} + \bar{A}_2 \frac{\beta^4 \eta^4}{\xi^4} \sqrt{\lambda^2 \xi^2 - \eta^2} \right] \\ \bar{G}_1 &= \xi^2 \left[ \bar{A}_3 + \bar{A}_4 \frac{\beta^2 \eta^2}{\xi^2} + \bar{A}_5 \frac{\beta^4 \eta^4}{\xi^4} \sqrt{\lambda^2 \xi^2 - \eta^2} \right] \end{aligned} \right\} \quad (16)$$

The coefficients  $A_i$  and  $\bar{A}_i$  in these equations are functions of  $\rho_0 = \beta\lambda$  only, where  $\lambda$  is the tangent of the half-apex angle of the triangle (see fig. 1). These coefficients arise in the process of solving the integral equations and are defined by systems of linear algebraic equations which are given in the appendix.

The integral equations for the distribution functions associated with  $\phi_B$ ,  $\phi_C$ , . . . , equation (5), are of the same general form as those associated with  $\phi_A$ . The main difference is that the term on the right side of equation (13) is replaced by the term appropriate to the downwash under consideration. The solutions yield the following distribution functions:

For  $\phi_B$ :

$$\left. \begin{aligned}
 G_0 &= \xi^2 \left[ B_0 + B_1 \frac{\beta^2 \eta^2}{\xi^2} \right] \sqrt{\lambda^2 \xi^2 - \eta^2} \\
 G_1 &= \xi^2 \left[ B_2 + B_3 \frac{\beta^2 \eta^2}{\xi^2} \right] \sqrt{\lambda^2 \xi^2 - \eta^2} \\
 G_2 &= \xi^2 \left[ B_4 + B_5 \frac{\beta^2 \eta^2}{\xi^2} + B_6 \frac{\beta^4 \eta^4}{\xi^4} \right] \sqrt{\lambda^2 \xi^2 - \eta^2} \\
 G_3 &= \xi^2 \left[ B_7 + B_8 \frac{\beta^2 \eta^2}{\xi^2} + B_9 \frac{\beta^4 \eta^4}{\xi^4} \right] \sqrt{\lambda^2 \xi^2 - \eta^2} \\
 \bar{G}_0 &= \xi^2 \left[ \bar{B}_0 + \bar{B}_1 \frac{\beta^2 \eta^2}{\xi^2} + \bar{B}_2 \frac{\beta^4 \eta^4}{\xi^4} \right] \sqrt{\lambda^2 \xi^2 - \eta^2} \\
 \bar{G}_1 &= \xi^2 \left[ \bar{B}_3 + \bar{B}_4 \frac{\beta^2 \eta^2}{\xi^2} + \bar{B}_5 \frac{\beta^4 \eta^4}{\xi^4} \right] \sqrt{\lambda^2 \xi^2 - \eta^2}
 \end{aligned} \right\} \quad (17)$$

for  $\phi_C$ :

$$\left. \begin{aligned}
 G_0 &= \beta \eta \xi \left( C_0 \sqrt{\lambda^2 \xi^2 - \eta^2} \right) \\
 G_1 &= \beta \eta \xi \left[ C_1 + C_2 \frac{\beta^2 \eta^2}{\xi^2} \right] \sqrt{\lambda^2 \xi^2 - \eta^2} \\
 G_2 &= \beta \eta \xi \left[ C_3 + C_4 \frac{\beta^2 \eta^2}{\xi^2} \right] \sqrt{\lambda^2 \xi^2 - \eta^2} \\
 G_3 &= \beta \eta \xi \left[ C_5 + C_6 \frac{\beta^2 \eta^2}{\xi^2} + C_7 \frac{\beta^4 \eta^4}{\xi^4} \right] \sqrt{\lambda^2 \xi^2 - \eta^2} \\
 \bar{G}_0 &= \beta \eta \xi \left[ \bar{C}_0 + \bar{C}_1 \frac{\beta^2 \eta^2}{\xi^2} \right] \sqrt{\lambda^2 \xi^2 - \eta^2} \\
 \bar{G}_1 &= \beta \eta \xi \left[ \bar{C}_2 + \bar{C}_3 \frac{\beta^2 \eta^2}{\xi^2} + \bar{C}_4 \frac{\beta^4 \eta^4}{\xi^4} \right] \sqrt{\lambda^2 \xi^2 - \eta^2}
 \end{aligned} \right\} \quad (18)$$

for  $\phi_D$ :

$$\left. \begin{aligned}
 G_0 &= \xi \left( D_0 \sqrt{\lambda^2 \xi^2 - \eta^2} \right) \\
 G_1 &= \xi \left[ \left( D_1 + D_2 \frac{\beta^2 \eta^2}{\xi^2} \right) \sqrt{\lambda^2 \xi^2 - \eta^2} \right] \\
 G_2 &= \xi \left[ \left( D_3 + D_4 \frac{\beta^2 \eta^2}{\xi^2} \right) \sqrt{\lambda^2 \xi^2 - \eta^2} \right] \\
 G_3 &= \xi \left[ \left( D_5 + D_6 \frac{\beta^2 \eta^2}{\xi^2} + D_7 \frac{\beta^4 \eta^4}{\xi^4} \right) \sqrt{\lambda^2 \xi^2 - \eta^2} \right] \\
 \bar{G}_0 &= \xi \left[ \left( \bar{D}_0 + \bar{D}_1 \frac{\beta^2 \eta^2}{\xi^2} \right) \sqrt{\lambda^2 \xi^2 - \eta^2} \right] \\
 \bar{G}_1 &= \xi \left[ \left( \bar{D}_2 + \bar{D}_3 \frac{\beta^2 \eta^2}{\xi^2} + \bar{D}_4 \frac{\beta^4 \eta^4}{\xi^4} \right) \sqrt{\lambda^2 \xi^2 - \eta^2} \right]
 \end{aligned} \right\} \quad (19)$$

for  $\phi_E$ :

$$\left. \begin{aligned}
 G_0 &= \beta \eta \left( E_0 \sqrt{\lambda^2 \xi^2 - \eta^2} \right) \\
 G_1 &= \beta \eta \left( E_1 \sqrt{\lambda^2 \xi^2 - \eta^2} \right) \\
 G_2 &= \beta \eta \left[ \left( E_2 + E_3 \frac{\beta^2 \eta^2}{\xi^2} \right) \sqrt{\lambda^2 \xi^2 - \eta^2} \right] \\
 G_3 &= \beta \eta \left[ \left( E_4 + E_5 \frac{\beta^2 \eta^2}{\xi^2} \right) \sqrt{\lambda^2 \xi^2 - \eta^2} \right] \\
 \bar{G}_0 &= \beta \eta \left[ \left( \bar{E}_0 + \bar{E}_1 \frac{\beta^2 \eta^2}{\xi^2} \right) \sqrt{\lambda^2 \xi^2 - \eta^2} \right] \\
 \bar{G}_1 &= \beta \eta \left[ \left( \bar{E}_2 + \bar{E}_3 \frac{\beta^2 \eta^2}{\xi^2} \right) \sqrt{\lambda^2 \xi^2 - \eta^2} \right]
 \end{aligned} \right\} \quad (20)$$

and, finally, for  $\phi_F$ :

$$\left. \begin{aligned} G_0 &= \left( F_0 \sqrt{\lambda^2 \xi^2 - \eta^2} \right) \\ G_1 &= \left( F_1 \sqrt{\lambda^2 \xi^2 - \eta^2} \right) \\ G_2 &= \left[ \left( F_2 + F_3 \frac{\beta^2 \eta^2}{\xi^2} \right) \sqrt{\lambda^2 \xi^2 - \eta^2} \right] \\ G_3 &= \left[ \left( F_4 + F_5 \frac{\beta^2 \eta^2}{\xi^2} \right) \sqrt{\lambda^2 \xi^2 - \eta^2} \right] \\ \bar{G}_0 &= \left[ \left( \bar{F}_0 + \bar{F}_1 \frac{\beta^2 \eta^2}{\xi^2} \right) \sqrt{\lambda^2 \xi^2 - \eta^2} \right] \\ \bar{G}_1 &= \left[ \left( \bar{F}_2 + \bar{F}_3 \frac{\beta^2 \eta^2}{\xi^2} \right) \sqrt{\lambda^2 \xi^2 - \eta^2} \right] \end{aligned} \right\} \quad (21)$$

The coefficients  $B_1, \bar{B}_1, C_1, \bar{C}_1, \dots$  are functions of  $\rho_0 = \beta\lambda$ . They are defined by linear simultaneous equations given in the appendix following those defining  $A_1, \bar{A}_1$ .

Expressions for the potential.— With the use of the distribution functions, the potentials can be written by inspection and comparison. For example, from equations (7) and (11) the expression for  $\phi_A$  in terms of  $G_n$  and  $\bar{G}_n$  is

$$\phi_A = Ae^{i\omega t} \sum_{n=0}^3 a_{n1} x^n G_n(x, y) + Ae^{i\omega t} \sum_{n=0}^1 \frac{\bar{\omega}^2}{2M^2} a_{n1} x^{n+2} \bar{G}_n(x, y) \quad (22)$$

The potentials  $\phi_B, \phi_C, \dots$  are represented by similar expressions. After terms have been combined with respect to  $\bar{\omega}$ , the subsidiary potentials to the third power of  $\bar{\omega}$  can be written in the following forms:

$$\phi_A = Ae^{i\omega t} \sqrt{\lambda^2 x^2 - y^2} \left[ a_0 x^2 + a_1 \beta^2 y^2 - i\bar{\omega} (a_2 x^3 + a_3 x \beta^2 y^2) - \bar{\omega}^2 (a_4 x^4 + a_5 x^2 \beta^2 y^2 + a_6 \beta^4 y^4) + i\bar{\omega}^3 (a_7 x^5 + a_8 x^3 \beta^2 y^2 + a_9 x \beta^4 y^4) \right] \quad (23)$$

$$\phi_B = Be^{i\omega t} \sqrt{\lambda^2 x^2 - y^2} \left[ b_0 x^2 + b_1 \beta^2 y^2 - i\bar{\omega} (b_2 x^3 + b_3 x \beta^2 y^2) - \bar{\omega}^2 (b_4 x^4 + b_5 x^2 \beta^2 y^2 + b_6 \beta^4 y^4) + i\bar{\omega}^3 (b_7 x^5 + b_8 x^3 \beta^2 y^2 + b_9 x \beta^4 y^4) \right] \quad (24)$$

$$\phi_C = Cye^{i\omega t} \sqrt{\lambda^2 x^2 - y^2} \left[ c_0 x - i\bar{\omega} (c_1 x^2 + c_2 \beta^2 y^2) - \bar{\omega}^2 (c_3 x^3 + c_4 x \beta^2 y^2) + i\bar{\omega}^3 (c_5 x^4 + c_6 x^2 \beta^2 y^2 + c_7 \beta^4 y^4) \right] \quad (25)$$

$$\phi_D = De^{i\omega t} \sqrt{\lambda^2 x^2 - y^2} \left[ d_0 x - i\bar{\omega} (d_1 x^2 + d_2 \beta^2 y^2) - \bar{\omega}^2 (d_3 x^3 + d_4 x \beta^2 y^2) + i\bar{\omega}^3 (d_5 x^4 + d_6 x^2 \beta^2 y^2 + d_7 \beta^4 y^4) \right] \quad (26)$$

$$\phi_E = Eye^{i\omega t} \sqrt{\lambda^2 x^2 - y^2} \left[ e_0 - i\bar{\omega} e_1 x - \bar{\omega}^2 (e_2 x^2 + e_3 \beta^2 y^2) + i\bar{\omega}^3 (e_4 x^3 + e_5 x \beta^2 y^2) \right] \quad (27)$$

$$\phi_F = Fe^{i\omega t} \sqrt{\lambda^2 x^2 - y^2} \left[ f_0 - i\bar{\omega} f_1 x - \bar{\omega}^2 (f_2 x^2 + f_3 \beta^2 y^2) + i\bar{\omega}^3 (f_4 x^3 + f_5 x \beta^2 y^2) \right] \quad (28)$$

In these equations  $a_1, b_1, \dots$  are linear combinations of  $A_1, \bar{A}_1, B_1, \bar{B}_1, \dots$  and can be further expressed as

$$a_1 = a_1' + \frac{a_1''}{M^2}$$

$$b_1 = b_1' + \frac{b_1''}{M^2}$$

. . . . .

Expressions for the quantities  $a_1', a_1'', b_1', b_1'', \dots$  are given in the appendix in terms of the quantities  $A_1, \bar{A}_1, B_1, \bar{B}_1, \dots$ . In addition, values for these quantities for particular values of  $\rho_0$  throughout the range  $0 \leq \rho_0 \leq 1$  are given in table I, and, for convenience in interpolation, they are shown plotted in figure 2. Thus, for a given triangular wing and a given Mach number, such that  $\rho_0^2 = \beta^2 \lambda^2 \leq 1$ , the coefficients  $a_1, b_1, \dots, f_1$  in equations (23) to (28) can be assigned their numerical values with the use of table I and figure 2.

It may be noted from the boundary condition, equation (4), that the subsidiary potentials  $\phi_D, \phi_E$ , and  $\phi_F$  are associated with motions of rigid wings; and that the potentials  $\phi_D$  and  $\phi_F$ , equations (26) and (28), could be obtained directly from reference 4 by comparison with the potentials denoted therein by  $\phi_\beta$  and  $\phi_\alpha$ , respectively. Also the parameter  $e_0$  in equation (27) agrees, as it should, with the parameter associated with constant rolling motion of a triangular wing (see, for example, ref. 7).

The sonic case.— At  $\beta\lambda = 1$  or  $\lambda = 1/\beta$ , which is the condition at which the Mach lines from the apex of the triangle coincide with the leading edges of the triangle, equations (23) to (28) reduce, respectively, to



$$\begin{aligned}
(\phi_A)_{\beta\lambda=1} = & \frac{2A\sqrt{x^2 - \beta^2 y^2}}{\beta\pi} e^{i\omega t} \left\{ \frac{2}{45}(11x^2 + \beta^2 y^2) - \frac{2i\bar{\omega}}{315}(17x^3 - 5x\beta^2 y^2) - \right. \\
& \frac{\bar{\omega}^2}{1575M^2} \left[ (17 + 29M^2)x^4 - (14 + 13M^2)x^2\beta^2 y^2 - (3 - 4M^2)\beta^4 y^4 \right] + \\
& \frac{i\bar{\omega}^3}{155925M^2} \left[ (759 + 401M^2)x^5 - (858 + 137M^2)x^3\beta^2 y^2 + \right. \\
& \left. \left. (99 + 36M^2)x\beta^4 y^4 \right] \right\} \quad (29)
\end{aligned}$$

$$\begin{aligned}
(\phi_B)_{\beta\lambda=1} = & \frac{2B\sqrt{x^2 - \beta^2 y^2}}{\beta\pi} e^{i\omega t} \left\{ \frac{2}{45}(x^2 + 11\beta^2 y^2) - \frac{2i\bar{\omega}}{105}(x^3 + 3x\beta^2 y^2) - \right. \\
& \frac{\bar{\omega}^2}{1575M^2} \left[ (3 + 7M^2)x^4 + (14 + M^2)x^2\beta^2 y^2 - (17 - 12M^2)\beta^4 y^4 \right] + \\
& \frac{i\bar{\omega}^3}{31185M^2} \left[ (33 + 23M^2)x^5 + (66 - 23M^2)x^3\beta^2 y^2 - \right. \\
& \left. \left. (99 - 60M^2)x\beta^4 y^4 \right] \right\} \quad (30)
\end{aligned}$$

$$\begin{aligned}
(\phi_C)_{\beta\lambda=1} = & \frac{2Cy\sqrt{x^2 - \beta^2 y^2}}{\beta\pi} e^{i\omega t} \left\{ \frac{8x}{15} - \frac{8i\bar{\omega}}{315}(4x^2 - \beta^2 y^2) - \right. \\
& \frac{4\bar{\omega}^2}{315M^2} \left[ (1 + M^2)x^3 - x\beta^2 y^2 \right] + \frac{4i\bar{\omega}^3}{51975M^2} \left[ (66 + 14M^2)x^4 - \right. \\
& \left. \left. (77 - 17M^2)x^2\beta^2 y^2 + (11 - 6M^2)\beta^4 y^4 \right] \right\} \quad (31)
\end{aligned}$$

$$\begin{aligned}
 (\phi_D)_{\beta\lambda=1} = \frac{2D\sqrt{x^2 - \beta^2 y^2}}{\beta\pi} e^{i\omega t} & \left\{ \frac{2x}{3} - \frac{2i\bar{\omega}}{45}(4x^2 - \beta^2 y^2) - \frac{\bar{\omega}^2}{315M^2} \left[ (7 + 11M^2)x^3 - \right. \right. \\
 & \left. (7 + 2M^2)x\beta^2 y^2 \right] + \frac{i\bar{\omega}^3}{4725M^2} \left[ (54 + 26M^2)x^4 - \right. \\
 & \left. \left. (63 - 3M^2)x^2\beta^2 y^2 + (9 - 4M^2)\beta^4 y^4 \right] \right\} \quad (32)
 \end{aligned}$$

$$\begin{aligned}
 (\phi_E)_{\beta\lambda=1} = \frac{2Ey\sqrt{x^2 - \beta^2 y^2}}{\beta\pi} e^{i\omega t} & \left\{ \frac{2}{3} - \frac{2i\bar{\omega}x}{15} - \frac{\bar{\omega}^2}{315M^2} \left[ (7 + 5M^2)x^2 - \right. \right. \\
 & \left. (7 - 4M^2)\beta^2 y^2 \right] + \frac{i\bar{\omega}^3}{945M^2} \left[ (9 + M^2)x^3 - (9 - 4M^2)x\beta^2 y^2 \right] \right\} \quad (33)
 \end{aligned}$$

$$\begin{aligned}
 (\phi_F)_{\beta\lambda=1} = \frac{2Fy\sqrt{x^2 - \beta^2 y^2}}{\beta\pi} e^{i\omega t} & \left\{ 1 - \frac{i\bar{\omega}x}{3} - \frac{\bar{\omega}^2}{90M^2} \left[ (5 + 7M^2)x^2 - \right. \right. \\
 & \left. (5 - 2M^2)\beta^2 y^2 \right] + \frac{i\bar{\omega}^3}{210M^2} \left[ (7 + 3M^2)x^3 - (7 - 2M^2)x\beta^2 y^2 \right] \right\} \quad (34)
 \end{aligned}$$

Equations (29) to (34) have also been obtained by integrating the expanded potentials of appropriate sources over the region occupied by the wing for this case of  $\beta\lambda = 1$ . They therefore serve as a check on the results given in equations (23) to (28).

Introduction of the reduced-frequency parameter into the expressions for the potentials.— In applications it is usually desirable to refer all lengths to some convenient reference length on the wing and to introduce a reduced-frequency parameter. For this reason it may be desirable to write the potentials in a slightly modified form. If, in the present case, the root chord  $2b$  is chosen as a reference length and the variables  $x$  and  $y$  in the potentials are employed in a new sense to mean that they are referred to this length, the expressions for the potentials

can be written in terms of the reduced-frequency parameter  $k = b\omega/v$  as follows:

$$\begin{aligned} \phi_A = 2b(4b^2A)e^{i\omega t} \sqrt{\lambda^2x^2 - y^2} & \left[ a_0x^2 + a_1\beta^2y^2 - \frac{2iM^2k}{\beta^2}(a_2x^3 + a_3x\beta^2y^2) - \right. \\ & \frac{4M^4k^2}{\beta^4}(a_4x^4 + a_5x^2\beta^2y^2 + a_6\beta^4y^4) + \\ & \left. \frac{8iM^6k^3}{\beta^6}(a_7x^5 + a_8x^3\beta^2y^2 + a_9x\beta^4y^4) \right] \end{aligned} \quad (35)$$

$$\begin{aligned} \phi_B = 2b(4b^2B)e^{i\omega t} \sqrt{\lambda^2x^2 - y^2} & \left[ b_0x^2 + b_1\beta^2y^2 - \frac{2iM^2k}{\beta^2}(b_2x^3 + b_3x\beta^2y^2) - \right. \\ & \frac{4M^4k^2}{\beta^4}(b_4x^4 + b_5x^2\beta^2y^2 + b_6\beta^4y^4) + \\ & \left. \frac{8iM^6k^3}{\beta^6}(b_7x^5 + b_8x^3\beta^2y^2 + b_9x\beta^4y^4) \right] \end{aligned} \quad (36)$$

$$\begin{aligned} \phi_C = 2b(4b^2C)ye^{i\omega t} \sqrt{\lambda^2x^2 - y^2} & \left[ c_0x - \frac{2iM^2k}{\beta^2}(c_1x^2 + c_2\beta^2y^2) - \right. \\ & \frac{4M^4k^2}{\beta^4}(c_3x^3 + c_4x\beta^2y^2) + \frac{8iM^6k^3}{\beta^6}(c_5x^4 + c_6x^2\beta^2y^2 + c_7\beta^4y^4) \left. \right] \end{aligned} \quad (37)$$

$$\begin{aligned} \phi_D = 2b(2bD)e^{i\omega t} \sqrt{\lambda^2x^2 - y^2} & \left[ d_0x - \frac{2iM^2k}{\beta^2}(d_1x^2 + d_2\beta^2y^2) - \right. \\ & \frac{4M^4k^2}{\beta^4}(d_3x^3 + d_4x\beta^2y^2) + \frac{8iM^6k^3}{\beta^6}(d_5x^4 + d_6x^2\beta^2y^2 + d_7\beta^4y^4) \left. \right] \end{aligned} \quad (38)$$

$$\phi_E = 2b(2bE)ye^{i\omega t} \sqrt{\lambda^2 x^2 - y^2} \left[ e_0 - \frac{2iM^2 k}{\beta^2} e_1 x - \frac{4M^4 k^2}{\beta^4} (e_2 x^2 + e_3 \beta^2 y^2) + \frac{8iM^6 k^3}{\beta^6} (e_4 x^3 + e_5 x \beta^2 y^2) \right] \quad (39)$$

$$\phi_F = 2b(F)e^{i\omega t} \sqrt{\lambda^2 x^2 - y^2} \left[ f_0 - \frac{2iM^2 k}{\beta^2} f_1 x - \frac{4M^4 k^2}{\beta^4} (f_2 x^2 + f_3 \beta^2 y^2) + \frac{8iM^6 k^3}{\beta^6} (f_4 x^3 + f_5 x \beta^2 y^2) \right] \quad (40)$$

(It is to be noted in these equations that the expressions within the first parentheses in each equation, namely those containing the coefficients A, B, C, . . . , are expressions for downwash and hence possess the dimensions of velocity.)

#### REMARKS AND DISCUSSION

The local force (positive downward) or pressure difference  $\Delta p$  between the upper and lower surface at any point (x,y) on the wing is, in terms of nondimensional coordinates,

$$\Delta p = -2\rho \left( \frac{V}{2b} \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial t} \right) \quad (41a)$$

or, for the oscillating case under consideration,

$$\Delta p = - \frac{\rho V}{b} \left( \frac{\partial}{\partial x} + 2ik \right) \phi \quad (41b)$$

By substituting the potentials given by equations (35) to (40) into these expressions, the pressure distributions can then be found for the types of motions or downwash considered in this paper. It should be pointed out that the expressions for potentials developed herein for a triangular wing can be used to calculate local pressures on certain other plan forms, as indicated in reference 4, which can be formed from the triangular wing by cutting the trailing edges in such a way that they lie ahead of Mach lines emanating from all points of the trailing edges.

The pressure distributions obtained in this way can then be integrated to obtain the desired spanwise distributions of the lift and moment, as well as any integrated quantities which may be of interest in the structural analysis of the given wing. For instance, in a modal analysis, for which the results of this paper would be particularly useful, the aerodynamic forces usually enter in the form of integrals over the wing of the product of local pressure difference and the local distortion in a given mode, these integrals representing virtual work done by the aerodynamic forces. The virtual-work integrals arising from any or all of the potentials presented in the foregoing analysis can be easily reduced to involve only integrations of functions of the form  $x^n(x^2 - a^2)^{\pm 1/2}$  and can therefore be readily evaluated. It is recognized that, if the wing root is considered to be rigidly fixed, some of the displacement terms considered herein will not satisfy boundary conditions at the wing root for either plate or beam theory. If, however, such terms are useful in approximating a known mode shape, these types of disparities may be overlooked with the interpretation that they imply large stresses at the wing root especially since with triangular wings such stresses are known to exist.

Inasmuch as the types of integrals required vary for different methods of aeroelastic analysis (depending on whether the theory for a simple beam or that for a more refined representative structure is used) and, inasmuch as the limits of these integrals depend on the exact plan form, further manipulations employing equations (41) to obtain some form of force and moment coefficients are not presented herein. However, in order to illustrate the results obtainable by the method of this paper, the spanwise variations of the unsteady-lift derivatives associated with parabolic bending (obtained from  $\phi_B$ ) are shown in figure 3 for a  $45^\circ$  delta wing undergoing parabolic bending at  $k = 0.1$  and at a Mach number of 1.2. These results may be used to give some indication of the way coefficients based on the true downwash conditions for the distorting wing compare with those obtainable by a simple strip-theory analysis based on the coefficients associated with rigid-wing motions. In order to effect such a comparison, the results of multiplying the lift derivatives associated with translation (obtained from  $\phi_T$ ) by the ordinates of the assumed parabolic mode shape ( $y^2$ ) are also shown in

figure 3 for the same value of  $k$  and  $M$ . The main features to note in this comparison are the differences in magnitude of lift at the wing root and at the points of maximum lift. These differences are, of course, associated with induced effects due to bending. In order to determine the nature of these induced effects on calculated flutter speeds, further investigations are required.

Langley Aeronautical Laboratory,  
National Advisory Committee for Aeronautics,  
Langley Field, Va., June 4, 1953.

## APPENDIX

## COEFFICIENTS ASSOCIATED WITH THE VELOCITY POTENTIAL

This appendix contains definitions of the coefficients that have arisen in the foregoing analysis.

The coefficients denoted by  $A_1, \bar{A}_1, B_1, \bar{B}_1, \dots$  in equations (16) to (21) of the body of this paper are defined by systems of linear algebraic equations which arise in the process employed for solving the integral equations involved. These algebraic equations are listed subsequently; however, they in turn involve a set of cumbersome coefficients that are denoted by  $W_{n,m}^P$  and  $\bar{W}_{n,m}^P$ . These coefficients are defined as follows:

Application of the transformations (see appendix of ref. 4)

$$N = \frac{\beta x(\theta - \sigma)}{1 - \beta^2 \sigma^2}$$

$$q = \frac{x(1 - \beta^2 \sigma \theta)}{1 - \beta^2 \sigma^2}$$

$$\theta = \frac{y}{x}$$

$$\sigma = \frac{\eta}{\xi}$$

to the integral equations (and certain derivatives thereof) which define the distribution functions (see eqs. (13) and (14) for  $\phi_A$ ) results in the definitions

$$W_{n,m}^p = - \frac{(n+2)\beta^2}{\pi x^n} \int_{-\lambda}^{+\lambda} \frac{\sigma^m \sqrt{\lambda^2 - \sigma^2}}{(1 - \beta^2 \sigma^2)^{3/2}} d\sigma \lim_{\theta \rightarrow 0} \frac{\partial^p}{\partial \theta^p} \int_0^{\cosh^{-1} \frac{q}{N}} N^n \left( \frac{q}{N} - \cosh \tau \right)^{n+1} \cosh \tau d\tau$$

$$\bar{W}_{n,m}^p = \frac{\beta^2}{\pi x^{n+1}} \int_{-\lambda}^{+\lambda} \frac{\sigma^m \sqrt{\lambda^2 - \sigma^2}}{\sqrt{1 - \beta^2 \sigma^2}} d\sigma \lim_{\theta \rightarrow 0} \frac{\partial^p}{\partial \theta^p} \int_0^{\cosh^{-1} \frac{q}{N}} (q - N \cosh \tau)^{n+1} \left[ 2q - (n+4)N \cosh \tau \right] \sinh^2 \tau d\tau$$

(Unless  $m$  and  $p$  are either both even or both odd,  $W_{n,m}^p$  and  $\bar{W}_{n,m}^p$  vanish owing to symmetry.)

With  $F'$  and  $E'$  denoting complete elliptic integrals of the first and second kinds, respectively, with modulus  $\sqrt{1 - \rho_0^2}$ , explicit values of  $W_{n,m}^p$  and  $\bar{W}_{n,m}^p$  required to write the equations for  $A_1, \bar{A}_1, B_1, \bar{B}_1, \dots$  are

$$W_{0,0}^0 = E'$$

$$W_{1,0}^0 = \frac{1}{1 - \rho_0^2} \left[ \rho_0^2 F' + (1 - 2\rho_0^2) E' \right]$$

$$W_{1,1}^1 = \frac{1}{1 - \rho_0^2} \left[ -\rho_0^2 F' + (2 - \rho_0^2) E' \right]$$



$$W_{2,0}^0 = \frac{1}{2(1 - \rho_0^2)^2} \left[ (5\rho_0^2 - 3\rho_0^4)F' + (2 - 10\rho_0^2 + 6\rho_0^4)E' \right]$$

$$W_{2,0}^2 = \frac{\beta^2}{(1 - \rho_0^2)^2} \left[ 2\rho_0^2 F' - (1 + \rho_0^2)E' \right]$$

$$W_{2,1}^1 = \frac{1}{(1 - \rho_0^2)^2} \left[ -(\rho_0^2 + \rho_0^4)F' + (2 - 2\rho_0^2 + 2\rho_0^4)E' \right]$$

$$W_{2,2}^0 = \frac{\rho_0^2}{2\beta^4} W_{2,0}^2$$

$$W_{2,2}^2 = \frac{1}{(1 - \rho_0^2)^2} \left[ (5\rho_0^4 - 3\rho_0^2)F' + (6 - 10\rho_0^2 + 2\rho_0^4)E' \right]$$

$$W_{3,0}^0 = \frac{1}{6(1 - \rho_0^2)^3} \left[ (27\rho_0^2 - 31\rho_0^4 + 12\rho_0^6)F' + (6 - 55\rho_0^2 + 65\rho_0^4 - 24\rho_0^6)E' \right]$$

$$W_{3,0}^2 = \frac{\beta^2}{(1 - \rho_0^2)^3} \left[ (9\rho_0^2 - \rho_0^4)F' - (3 + 7\rho_0^2 - 2\rho_0^4)E' \right]$$

$$W_{3,1}^1 = \frac{1}{2(1 - \rho_0^2)^3} \left[ (3\rho_0^6 - 9\rho_0^4 - 2\rho_0^2)F' + (4 - 5\rho_0^2 + 15\rho_0^4 - 6\rho_0^6)E' \right]$$

$$w_{3,1}^3 = \frac{\beta^2}{(1 - \rho_0^2)^3} \left[ (\rho_0^2 - 9\rho_0^4)_{F'} - (2 - 7\rho_0^2 - 3\rho_0^4)_{E'} \right]$$

$$w_{3,2}^0 = \frac{\rho_0^2}{6\beta^4} w_{3,0}^2$$

$$w_{3,2}^2 = \frac{1}{(1 - \rho_0^2)^3} \left[ (2\rho_0^6 + 9\rho_0^4 - 3\rho_0^2)_{F'} + (6 - 15\rho_0^2 + 5\rho_0^4 - 4\rho_0^6)_{E'} \right]$$

$$w_{3,3}^1 = \frac{\rho_0^2}{2\beta^4} w_{3,1}^3$$

$$w_{3,3}^3 = \frac{1}{(1 - \rho_0^2)^3} \left[ -(12\rho_0^2 - 31\rho_0^4 + 27\rho_0^6)_{F'} + (24 - 65\rho_0^2 + 55\rho_0^4 - 6\rho_0^6)_{E'} \right]$$

$$w_{4,0}^0 = \frac{1}{8(1 - \rho_0^2)^4} \left[ (56\rho_0^2 - 92\rho_0^4 + 72\rho_0^6 - 20\rho_0^8)_{F'} + (8 - 117\rho_0^2 + 202\rho_0^4 - 149\rho_0^6 + 40\rho_0^8)_{E'} \right]$$

$$w_{4,0}^2 = \frac{3\beta^2}{2(1 - \rho_0^2)^4} \left[ (17\rho_0^2 - 2\rho_0^4 + \rho_0^6)_{F'} - (4 + 18\rho_0^2 - 8\rho_0^4 + 2\rho_0^6)_{E'} \right]$$

$$w_{4,0}^4 = \frac{3\beta^4}{(1 - \rho_0^2)^4} \left[ (8\rho_0^2 + 8\rho_0^4)F' - (1 + 14\rho_0^2 + \rho_0^4)E' \right]$$

$$w_{4,1}^1 = \frac{1}{2(1 - \rho_0^2)^4} \left[ -(2\rho_0^2 + 24\rho_0^4 - 14\rho_0^6 + 4\rho_0^8)F' + (4 - 5\rho_0^2 + 38\rho_0^4 - 29\rho_0^6 + 8\rho_0^8)E' \right]$$

$$w_{4,1}^3 = \frac{3\beta^2}{(1 - \rho_0^2)^4} \left[ (\rho_0^2 - 18\rho_0^4 + \rho_0^6)F' - (2 - 10\rho_0^2 - 10\rho_0^4 + 2\rho_0^6)E' \right]$$

$$w_{4,2}^0 = \frac{\rho_0^2}{12\beta^4} w_{4,0}^2$$

$$w_{4,2}^2 = \frac{3}{2(1 - \rho_0^2)^4} \left[ -(2\rho_0^2 - 10\rho_0^4 - 10\rho_0^6 + 2\rho_0^8)F' + (4 - 13\rho_0^2 + 2\rho_0^4 - 13\rho_0^6 + 4\rho_0^8)E' \right]$$

$$w_{4,2}^4 = \frac{3\beta^2}{(1 - \rho_0^2)^4} \left[ (\rho_0^2 - 2\rho_0^4 + 17\rho_0^6)F' - (2 - 8\rho_0^2 + 18\rho_0^4 + 4\rho_0^6)E' \right]$$

$$w_{4,3}^1 = \frac{\rho_0^2}{6\beta^4} w_{4,1}^3$$

$$W_{4,3}^3 = \frac{1}{(1 - \rho_0^2)^4} \left[ -(12\rho_0^2 - 42\rho_0^4 + 72\rho_0^6 + 6\rho_0^8)F' + (24 - 87\rho_0^2 + 114\rho_0^4 - 15\rho_0^6 + 12\rho_0^8)E' \right]$$

$$W_{4,4}^0 = \frac{\rho_0^4}{24\beta^8} W_{4,0}^4$$

$$W_{4,4}^2 = \frac{3}{2\beta^2(1 - \rho_0^2)^4} \left[ (\rho_0^4 - 2\rho_0^6 + 17\rho_0^8)F' - (2\rho_0^2 - 8\rho_0^4 + 18\rho_0^6 + 4\rho_0^8)E' \right]$$

$$W_{4,4}^4 = \frac{3}{(1 - \rho_0^2)^4} \left[ (56\rho_0^8 - 92\rho_0^6 + 72\rho_0^4 - 20\rho_0^2)F' + (40 - 149\rho_0^2 + 202\rho_0^4 - 117\rho_0^6 + 8\rho_0^8)E' \right]$$

$$W_{5,0}^0 = \frac{1}{40(1 - \rho_0^2)^5} \left[ (400\rho_0^2 - 833\rho_0^4 + 994\rho_0^6 - 553\rho_0^8 + 120\rho_0^{10})F' + (40 - 859\rho_0^2 + 1910\rho_0^4 - 2115\rho_0^6 + 1136\rho_0^8 - 240\rho_0^{10})E' \right]$$

$$W_{5,0}^2 = \frac{\beta^2}{2(1 - \rho_0^2)^5} \left[ (115\rho_0^2 - 2\rho_0^4 + 19\rho_0^6 - 4\rho_0^8)F' - (20 + 151\rho_0^2 - 74\rho_0^4 + 39\rho_0^6 - 8\rho_0^8)E' \right]$$

$$w_{5,0}^4 = \frac{3\beta^4}{(1 - \rho_0^2)^5} \left[ (55\rho_0^2 + 74\rho_0^4 - \rho_0^6)F' - (5 + 108\rho_0^2 + 17\rho_0^4 - 2\rho_0^6)E' \right]$$

$$w_{5,1}^1 = \frac{1}{8(1 - \rho_0^2)^5} \left[ (20\rho_0^{10} - 91\rho_0^8 + 154\rho_0^6 - 203\rho_0^4 - 8\rho_0^2)F' + \right. \\ \left. (16 - 16\rho_0^2 + 323\rho_0^4 - 342\rho_0^6 + 187\rho_0^8 - 40\rho_0^{10})E' \right]$$

$$w_{5,1}^3 = \frac{3\beta^2}{2(1 - \rho_0^2)^5} \left[ (4\rho_0^2 - 131\rho_0^4 + 2\rho_0^6 - 3\rho_0^8)F' - (8 - 53\rho_0^2 - \right. \\ \left. 108\rho_0^4 + 31\rho_0^6 - 6\rho_0^8)E' \right]$$

$$w_{5,1}^5 = \frac{3\beta^4}{(1 - \rho_0^2)^5} \left[ (\rho_0^2 - 74\rho_0^4 - 55\rho_0^6)F' - (2 - 17\rho_0^2 - 108\rho_0^4 - 5\rho_0^6)E' \right]$$

$$w_{5,2}^0 = \frac{\rho_0^2}{20\beta^4} w_{5,0}^2$$

$$w_{5,2}^2 = \frac{1}{2(1 - \rho_0^2)^5} \left[ (8\rho_0^{10} - 35\rho_0^8 + 112\rho_0^6 + 49\rho_0^4 - 6\rho_0^2)F' + \right. \\ \left. (12 - 47\rho_0^2 - 22\rho_0^4 - 127\rho_0^6 + 72\rho_0^8 - 16\rho_0^{10})E' \right]$$

$$w_{5,2}^4 = \frac{3\beta^2}{(1 - \rho_0^2)^5} \left[ (3\rho_0^2 - 2\rho_0^4 + 131\rho_0^6 - 4\rho_0^8)F' - (6 - 31\rho_0^2 + \right. \\ \left. 108\rho_0^4 + 53\rho_0^6 - 8\rho_0^8)E' \right]$$

$$W_{5,3}^1 = \frac{\rho_0^2}{12\beta^4} W_{5,1}^3$$

$$W_{5,3}^3 = \frac{3}{2(1 - \rho_0^2)^5} \left[ (6\rho_0^{10} - 49\rho_0^8 - 112\rho_0^6 + 35\rho_0^4 - 8\rho_0^2)F' + \right. \\ \left. (16 - 72\rho_0^2 + 127\rho_0^4 + 22\rho_0^6 + 47\rho_0^8 - 12\rho_0^{10})E' \right]$$

$$W_{5,3}^5 = \frac{3\beta^2}{(1 - \rho_0^2)^5} \left[ (4\rho_0^2 - 19\rho_0^4 + 2\rho_0^6 - 115\rho_0^8)F' - (8 - 39\rho_0^2 + \right. \\ \left. 74\rho_0^4 - 151\rho_0^6 - 20\rho_0^8)E' \right]$$

$$W_{5,4}^0 = \frac{\rho_0^4}{120\beta^8} W_{5,0}^4$$

$$W_{5,4}^2 = \frac{1}{2\beta^2(1 - \rho_0^2)^5} \left[ (3\rho_0^4 - 2\rho_0^6 + 131\rho_0^8 - 4\rho_0^{10})F' - (6\rho_0^2 - \right. \\ \left. 31\rho_0^4 + 108\rho_0^6 + 53\rho_0^8 - 8\rho_0^{10})E' \right]$$

$$W_{5,4}^4 = \frac{3}{(1 - \rho_0^2)^5} \left[ (8\rho_0^{10} + 203\rho_0^8 - 154\rho_0^6 + 91\rho_0^4 - 20\rho_0^2)F' + \right. \\ \left. (40 - 187\rho_0^2 + 342\rho_0^4 - 323\rho_0^6 + 16\rho_0^8 - 16\rho_0^{10})E' \right]$$

$$W_{5,5}^1 = \frac{\rho_0^4}{24\beta^8} W_{5,1}^5$$

$$W_{5,5}^3 = \frac{\rho_0^2}{2\beta^4} W_{5,3}^2$$

$$W_{5,5}^5 = \frac{3}{(1 - \rho_0^2)^5} \left[ -(120\rho_0^2 - 553\rho_0^4 + 994\rho_0^6 - 833\rho_0^8 + 400\rho_0^{10})F' + (240 - 1136\rho_0^2 + 2115\rho_0^4 - 1910\rho_0^6 + 859\rho_0^8 - 40\rho_0^{10})E' \right]$$

$$\bar{W}_{0,0}^0 = -\frac{1}{2(1 - \rho_0^2)} (\rho_0^2 F' - \rho_0^2 E')$$

$$\bar{W}_{0,0}^2 = -\frac{\beta^2}{1 - \rho_0^2} (\rho_0^2 F' - E')$$

$$\bar{W}_{1,0}^0 = -\frac{1}{6(1 - \rho_0^2)^2} \left[ (3\rho_0^2 - \rho_0^4)F' - (4\rho_0^2 - 2\rho_0^4)E' \right]$$

$$\bar{W}_{1,0}^2 = -W_{2,0}^2$$

$$\bar{W}_{1,1}^1 = \frac{\rho_0^2}{2\beta^2} W_{2,0}^2$$

$$\bar{W}_{1,1}^3 = -\frac{\beta^2}{(1 - \rho_0^2)^2} \left[ (\rho_0^2 - 3\rho_0^4)F' - (2 - 4\rho_0^2)E' \right]$$

$$\bar{W}_{2,0}^0 = -\frac{1}{24(1 - \rho_0^2)^3} \left[ (12\rho_0^2 - 7\rho_0^4 + 3\rho_0^6)F' - (19\rho_0^2 - 17\rho_0^4 + 6\rho_0^6)E' \right]$$

$$\bar{w}_{2,0}^2 = - \frac{\beta^2}{2(1 - \rho_0^2)^3} \left[ (7\rho_0^2 + \rho_0^4)F' - (2 + 7\rho_0^2 - \rho_0^4)E' \right]$$

$$\bar{w}_{2,0}^4 = - \frac{\beta^4}{(1 - \rho_0^2)^3} \left[ (5\rho_0^2 + 3\rho_0^4)F' - (1 + 7\rho_0^2)E' \right]$$

$$\bar{w}_{2,1}^1 = \beta^2 w_{3,2}^0$$

$$\bar{w}_{2,1}^3 = - \frac{\beta^2}{(1 - \rho_0^2)^3} \left[ (\rho_0^2 - 9\rho_0^4)F' - (2 - 7\rho_0^2 - 3\rho_0^4)E' \right]$$

$$\bar{w}_{2,2}^0 = - \frac{1}{24\beta^2(1 - \rho_0^2)^3} \left[ (3\rho_0^4 + 5\rho_0^6)F' - (7\rho_0^4 + \rho_0^6)E' \right]$$

$$\bar{w}_{2,2}^2 = - \frac{1}{2(1 - \rho_0^2)^3} \left[ (\rho_0^4 + 7\rho_0^6)F' + (\rho_0^2 - 7\rho_0^4 - 2\rho_0^6)E' \right]$$

$$\bar{w}_{2,2}^4 = - \frac{\beta^2}{(1 - \rho_0^2)^3} \left[ (3\rho_0^2 - 7\rho_0^4 + 12\rho_0^6)F' - (6 - 17\rho_0^2 + 19\rho_0^4)E' \right]$$

$$\bar{w}_{3,0}^0 = - \frac{1}{120(1 - \rho_0^2)^4} \left[ (60\rho_0^2 - 44\rho_0^4 + 44\rho_0^6 - 12\rho_0^8)F' - (107\rho_0^2 - 126\rho_0^4 + 91\rho_0^6 - 24\rho_0^8)E' \right]$$

$$\bar{w}_{3,0}^2 = - \frac{\beta^2}{6(1 - \rho_0^2)^4} \left[ (33\rho_0^2 + 14\rho_0^4 + \rho_0^6)F' - (6 + 46\rho_0^2 - 6\rho_0^4 + 2\rho_0^6)E' \right]$$



$$\overline{w}_{3,0}^4 = -w_{4,0}^4$$

$$\overline{w}_{3,1}^1 = \frac{1}{8(1 - \rho_0^2)^4} \left[ (17\rho_0^4 - 2\rho_0^6 + \rho_0^8)_{F'} - (4\rho_0^2 + 18\rho_0^4 - 8\rho_0^6 + 2\rho_0^8)_{E'} \right]$$

$$\overline{w}_{3,1}^3 = -\frac{\beta^2}{2(1 - \rho_0^2)^4} \left[ (2\rho_0^2 - 44\rho_0^4 - 6\rho_0^6)_{F'} - (4 - 21\rho_0^2 - 34\rho_0^4 + 3\rho_0^6)_{E'} \right]$$

$$\overline{w}_{3,1}^5 = -\frac{\beta^4}{(1 - \rho_0^2)^4} \left[ (\rho_0^2 - 34\rho_0^4 - 15\rho_0^6)_{F'} - (2 - 12\rho_0^2 - 38\rho_0^4)_{E'} \right]$$

$$\overline{w}_{3,2}^0 = -\frac{1}{120\beta^2(1 - \rho_0^2)^4} \left[ (15\rho_0^4 + 34\rho_0^6 - \rho_0^8)_{F'} - (38\rho_0^4 + 12\rho_0^6 - 2\rho_0^8)_{E'} \right]$$

$$\overline{w}_{3,2}^2 = -\frac{1}{6(1 - \rho_0^2)^4} \left[ (6\rho_0^4 + 44\rho_0^6 - 2\rho_0^8)_{F'} + (3\rho_0^2 - 34\rho_0^4 - 21\rho_0^6 + 4\rho_0^8)_{E'} \right]$$

$$\overline{w}_{3,2}^4 = -w_{4,2}^4$$

$$\overline{w}_{3,3}^1 = \frac{\rho_0^4}{24\beta^6} w_{4,0}^4$$

$$\overline{w}_{3,3}^3 = \frac{1}{2(1 - \rho_0^2)^4} \left[ (\rho_0^4 + 14\rho_0^6 + 33\rho_0^8)_{F'} - (2\rho_0^2 - 6\rho_0^4 + 46\rho_0^6 + 6\rho_0^8)_{E'} \right]$$

$$\bar{W}_{3,3} = - \frac{\beta^2}{(1 - \rho_0^2)^4} \left[ (12\rho_0^2 - 44\rho_0^4 + 44\rho_0^6 - 60\rho_0^8)F' - (24 - 91\rho_0^2 + 126\rho_0^4 - 107\rho_0^6)E' \right]$$

The necessary equations for  $A_i, \bar{A}_i, B_i, \bar{B}_i, \dots$  are, in terms of  $W_{n,m}^P$  and  $\bar{W}_{n,m}^P$ , as follows:

For  $\phi_A$ :

$$\left. \begin{aligned} A_0 W_{2,0}^0 + A_1 \beta^2 W_{2,2}^0 &= 1 \\ A_0 W_{2,0}^2 + A_1 \beta^2 W_{2,2}^2 &= 0 \end{aligned} \right\}$$

$$\left. \begin{aligned} A_2 W_{3,0}^0 + A_3 \beta^2 W_{3,2}^0 &= 1 \\ A_2 W_{3,0}^2 + A_3 \beta^2 W_{3,2}^2 &= 0 \end{aligned} \right\}$$

$$\left. \begin{aligned} A_4 W_{4,0}^0 + A_5 \beta^2 W_{4,2}^0 + A_6 \beta^4 W_{4,4}^0 &= 1 \\ A_4 W_{4,0}^2 + A_5 \beta^2 W_{4,2}^2 + A_6 \beta^4 W_{4,4}^2 &= 0 \\ A_4 W_{4,0}^4 + A_5 \beta^2 W_{4,2}^4 + A_6 \beta^4 W_{4,4}^4 &= 0 \end{aligned} \right\}$$

$$\left. \begin{aligned} A_7 W_{5,0}^0 + A_8 \beta^2 W_{5,2}^0 + A_9 \beta^4 W_{5,4}^0 &= 1 \\ A_7 W_{5,0}^2 + A_8 \beta^2 W_{5,2}^2 + A_9 \beta^4 W_{5,4}^2 &= 0 \\ A_7 W_{5,0}^4 + A_8 \beta^2 W_{5,2}^4 + A_9 \beta^4 W_{5,4}^4 &= 0 \end{aligned} \right\}$$

$$\left. \begin{aligned} \bar{A}_0 W_{4,0}^0 + \bar{A}_1 \beta^2 W_{4,2}^0 + \bar{A}_2 \beta^4 W_{4,4}^0 &= A_0 \bar{W}_{2,0}^0 + A_1 \beta^2 \bar{W}_{2,2}^0 \\ \bar{A}_0 W_{4,0}^2 + \bar{A}_1 \beta^2 W_{4,2}^2 + \bar{A}_2 \beta^4 W_{4,4}^2 &= A_0 \bar{W}_{2,0}^2 + A_1 \beta^2 \bar{W}_{2,2}^2 \\ \bar{A}_0 W_{4,0}^4 + \bar{A}_1 \beta^2 W_{4,2}^4 + \bar{A}_2 \beta^4 W_{4,4}^4 &= A_0 \bar{W}_{2,0}^4 + A_1 \beta^2 \bar{W}_{2,2}^4 \end{aligned} \right\}$$

$$\left. \begin{aligned} \bar{A}_3 W_{5,0}^0 + \bar{A}_4 \beta^2 W_{5,2}^0 + \bar{A}_5 \beta^4 W_{5,4}^0 &= A_2 \bar{W}_{3,0}^0 + A_3 \beta^2 \bar{W}_{3,2}^0 \\ \bar{A}_3 W_{5,0}^2 + \bar{A}_4 \beta^2 W_{5,2}^2 + \bar{A}_5 \beta^4 W_{5,4}^2 &= A_2 \bar{W}_{3,0}^2 + A_3 \beta^2 \bar{W}_{3,2}^2 \\ \bar{A}_3 W_{5,0}^4 + \bar{A}_4 \beta^2 W_{5,2}^4 + \bar{A}_5 \beta^4 W_{5,4}^4 &= A_2 \bar{W}_{3,0}^4 + A_3 \beta^2 \bar{W}_{3,2}^4 \end{aligned} \right\}$$

for  $\phi_B$ :

$$\left. \begin{aligned} B_0 W_{2,0}^0 + B_1 \beta^2 W_{2,2}^0 &= 0 \\ B_0 W_{2,0}^2 + B_1 \beta^2 W_{2,2}^2 &= 2\beta^2 \end{aligned} \right\}$$

$$\left. \begin{aligned} B_2 W_{3,0}^0 + B_3 \beta^2 W_{3,2}^0 &= 0 \\ B_2 W_{3,0}^2 + B_3 \beta^2 W_{3,2}^2 &= 2\beta^2 \end{aligned} \right\}$$

$$\left. \begin{aligned} B_4 W_{4,0}^0 + B_5 \beta^2 W_{4,2}^0 + B_6 \beta^4 W_{4,4}^0 &= 0 \\ B_4 W_{4,0}^2 + B_5 \beta^2 W_{4,2}^2 + B_6 \beta^4 W_{4,4}^2 &= 2\beta^2 \\ B_4 W_{4,0}^4 + B_5 \beta^2 W_{4,2}^4 + B_6 \beta^4 W_{4,4}^4 &= 0 \end{aligned} \right\}$$

$$\left. \begin{aligned} B_7 W_{5,0}^0 + B_8 \beta^2 W_{5,2}^0 + B_9 \beta^4 W_{5,4}^0 &= 0 \\ B_7 W_{5,0}^2 + B_8 \beta^2 W_{5,2}^2 + B_9 \beta^4 W_{5,4}^2 &= 2\beta^2 \\ B_7 W_{5,0}^4 + B_8 \beta^2 W_{5,2}^4 + B_9 \beta^4 W_{5,4}^4 &= 0 \end{aligned} \right\}$$

$$\left. \begin{aligned} \bar{B}_0 W_{4,0}^0 + \bar{B}_1 \beta^2 W_{4,2}^0 + \bar{B}_2 \beta^4 W_{4,4}^0 &= B_0 \bar{W}_{2,0}^0 + B_1 \beta^2 \bar{W}_{2,2}^0 \\ \bar{B}_0 W_{4,0}^2 + \bar{B}_1 \beta^2 W_{4,2}^2 + \bar{B}_2 \beta^4 W_{4,4}^2 &= B_0 \bar{W}_{2,0}^2 + B_1 \beta^2 \bar{W}_{2,2}^2 \\ \bar{B}_0 W_{4,0}^4 + \bar{B}_1 \beta^2 W_{4,2}^4 + \bar{B}_2 \beta^4 W_{4,4}^4 &= B_0 \bar{W}_{2,0}^4 + B_1 \beta^2 \bar{W}_{2,2}^4 \end{aligned} \right\}$$

$$\left. \begin{aligned} \bar{B}_3 W_{5,0}^0 + \bar{B}_4 \beta^2 W_{5,2}^0 + \bar{B}_5 \beta^4 W_{5,4}^0 &= B_2 \bar{W}_{3,0}^0 + B_3 \beta^2 \bar{W}_{3,2}^0 \\ \bar{B}_3 W_{5,0}^2 + \bar{B}_4 \beta^2 W_{5,2}^2 + \bar{B}_5 \beta^4 W_{5,4}^2 &= B_2 \bar{W}_{3,0}^2 + B_3 \beta^2 \bar{W}_{3,2}^2 \\ \bar{B}_3 W_{5,0}^4 + \bar{B}_4 \beta^2 W_{5,2}^4 + \bar{B}_5 \beta^4 W_{5,4}^4 &= B_2 \bar{W}_{3,0}^4 + B_3 \beta^2 \bar{W}_{3,2}^4 \end{aligned} \right\}$$

for  $\phi_C$ :

$$C_0 W_{2,1}^1 = 1$$

$$\left. \begin{aligned} C_1 W_{3,1}^1 + C_2 \beta^2 W_{3,3}^1 &= 1 \\ C_1 W_{3,1}^3 + C_2 \beta^2 W_{3,3}^3 &= 0 \end{aligned} \right\}$$

$$\left. \begin{aligned} C_3 W_{4,1}^1 + C_4 \beta^2 W_{4,3}^1 &= 1 \\ C_3 W_{4,1}^3 + C_4 \beta^2 W_{4,3}^3 &= 0 \end{aligned} \right\}$$

$$\left. \begin{aligned} C_5 W_{5,1}^1 + C_6 \beta W_{5,3}^1 + C_7 \beta^4 W_{5,5}^1 &= 1 \\ C_5 W_{5,1}^3 + C_6 \beta W_{5,3}^3 + C_7 \beta^4 W_{5,5}^3 &= 0 \\ C_5 W_{5,1}^5 + C_6 \beta W_{5,3}^5 + C_7 \beta^4 W_{5,5}^5 &= 0 \end{aligned} \right\}$$

$$\left. \begin{aligned} \bar{C}_0 W_{4,1}^1 + \bar{C}_1 \beta W_{4,3}^1 &= C_0 \bar{W}_{2,1}^1 \\ \bar{C}_0 W_{4,1}^3 + \bar{C}_1 \beta W_{4,3}^3 &= C_0 \bar{W}_{2,1}^3 \end{aligned} \right\}$$

$$\left. \begin{aligned} \bar{C}_2 W_{5,1}^1 + \bar{C}_3 \beta W_{5,3}^1 + \bar{C}_4 \beta^4 W_{5,5}^1 &= C_1 \bar{W}_{3,1}^1 + C_2 \beta \bar{W}_{3,3}^1 \\ \bar{C}_2 W_{5,1}^3 + \bar{C}_3 \beta W_{5,3}^3 + \bar{C}_4 \beta^4 W_{5,5}^3 &= C_1 \bar{W}_{3,1}^3 + C_2 \beta \bar{W}_{3,3}^3 \\ \bar{C}_2 W_{5,1}^5 + \bar{C}_3 \beta W_{5,3}^5 + \bar{C}_4 \beta^4 W_{5,5}^5 &= C_1 \bar{W}_{3,1}^5 + C_2 \beta \bar{W}_{3,3}^5 \end{aligned} \right\}$$

for  $\phi_D$ :

$$D_0 W_{1,0}^0 = 1$$

$$\left. \begin{aligned} D_1 W_{2,0}^0 + D_2 \beta W_{2,2}^0 &= 1 \\ D_1 W_{2,0}^2 + D_2 \beta W_{2,2}^2 &= 0 \end{aligned} \right\}$$

$$\left. \begin{aligned} D_3 W_{3,0}^0 + D_4 \beta W_{3,2}^0 &= 1 \\ D_3 W_{3,0}^2 + D_4 \beta W_{3,2}^2 &= 0 \end{aligned} \right\}$$

$$\left. \begin{aligned} D_5 W_{4,0}^0 + D_6 \beta^2 W_{4,2}^0 + D_7 \beta^4 W_{4,4}^0 &= 1 \\ D_5 W_{4,0}^2 + D_6 \beta^2 W_{4,2}^2 + D_7 \beta^4 W_{4,4}^2 &= 0 \\ D_5 W_{4,0}^4 + D_6 \beta^2 W_{4,2}^4 + D_7 \beta^4 W_{4,4}^4 &= 0 \end{aligned} \right\}$$

$$\left. \begin{aligned} \bar{D}_0 W_{3,0}^0 + \bar{D}_1 \beta^2 W_{3,2}^0 &= D_0 \bar{W}_{1,0}^0 \\ \bar{D}_0 W_{3,0}^2 + \bar{D}_1 \beta^2 W_{3,2}^2 &= D_0 \bar{W}_{1,0}^2 \end{aligned} \right\}$$

$$\left. \begin{aligned} \bar{D}_2 W_{4,0}^0 + \bar{D}_3 \beta^2 W_{4,2}^0 + \bar{D}_4 \beta^4 W_{4,4}^0 &= D_1 \bar{W}_{2,0}^0 + D_2 \beta^2 \bar{W}_{2,2}^0 \\ \bar{D}_2 W_{4,0}^2 + \bar{D}_3 \beta^2 W_{4,2}^2 + \bar{D}_4 \beta^4 W_{4,4}^2 &= D_1 \bar{W}_{2,0}^2 + D_2 \beta^2 \bar{W}_{2,2}^2 \\ \bar{D}_2 W_{4,0}^4 + \bar{D}_3 \beta^2 W_{4,2}^4 + \bar{D}_4 \beta^4 W_{4,4}^4 &= D_1 \bar{W}_{2,0}^4 + D_2 \beta^2 \bar{W}_{2,2}^4 \end{aligned} \right\}$$

for  $\phi_E$ :

$$E_0 W_{1,1}^1 = 1$$

$$E_1 W_{2,1}^1 = 1$$

$$\left. \begin{aligned} E_2 W_{3,1}^1 + E_3 \beta^2 W_{3,3}^1 &= 1 \\ E_2 W_{3,1}^3 + E_3 \beta^2 W_{3,3}^3 &= 0 \end{aligned} \right\}$$

$$\left. \begin{aligned} E_4 W_{4,1}^1 + E_5 \beta^2 W_{4,3}^1 &= 1 \\ E_4 W_{4,1}^3 + E_5 \beta^2 W_{4,3}^3 &= 0 \end{aligned} \right\}$$

$$\left. \begin{aligned} \bar{E}_0 W_{3,1}^1 + \bar{E}_1 \beta^2 W_{3,3}^1 &= E_0 \bar{W}_{1,1}^1 \\ \bar{E}_0 W_{3,1}^3 + \bar{E}_1 \beta^2 W_{3,3}^3 &= E_0 \bar{W}_{1,1}^3 \end{aligned} \right\}$$

$$\left. \begin{aligned} \bar{E}_2 W_{4,1}^1 + \bar{E}_3 \beta^2 W_{4,3}^1 &= E_1 \bar{W}_{2,1}^1 \\ \bar{E}_2 W_{4,1}^3 + \bar{E}_3 \beta^2 W_{4,3}^3 &= E_1 \bar{W}_{2,1}^3 \end{aligned} \right\}$$

for  $\phi_F$ :

$$F_0 W_{0,0}^0 = 1$$

$$F_1 W_{1,0}^0 = 1$$

$$\left. \begin{aligned} F_2 W_{2,0}^0 + F_3 \beta^2 W_{2,2}^0 &= 1 \\ F_2 W_{2,0}^2 + F_3 \beta^2 W_{2,2}^2 &= 0 \end{aligned} \right\}$$

$$\left. \begin{aligned} F_4 W_{3,0}^0 + F_5 \beta^2 W_{3,2}^0 &= 1 \\ F_4 W_{3,0}^2 + F_5 \beta^2 W_{3,2}^2 &= 0 \end{aligned} \right\}$$

$$\left. \begin{aligned} \bar{F}_0 W_{2,0}^0 + \bar{F}_1 \beta^2 W_{2,2}^0 &= F_0 \bar{W}_{0,0}^0 \\ \bar{F}_0 W_{2,0}^2 + \bar{F}_1 \beta^2 W_{2,2}^2 &= F_0 \bar{W}_{0,0}^2 \end{aligned} \right\}$$

$$\left. \begin{aligned} \bar{F}_2 W_{3,0}^0 + \bar{F}_3 \beta^2 W_{3,2}^0 &= F_1 \bar{W}_{1,0}^0 \\ \bar{F}_2 W_{3,0}^2 + \bar{F}_3 \beta^2 W_{3,2}^2 &= F_1 \bar{W}_{1,0}^2 \end{aligned} \right\}$$

The coefficients  $a_1'$ ,  $a_1''$ ,  $b_1'$ ,  $b_1''$ , . . . used in the expressions for the velocity potential (eqs. (23) to (28)) are defined in terms of  $A_1$ ,  $\bar{A}_1$ ,  $B_1$ ,  $\bar{B}_1$ , . . . as follows:

$$a_0' = A_0$$

$$a_0'' = 0$$

$$a_1' = A_1$$

$$a_1'' = 0$$

$$a_2' = A_0 - A_2$$

$$a_2'' = 0$$

$$a_3' = A_1 - A_3$$

$$a_3'' = 0$$

$$a_4' = \frac{A_0}{2} - A_2 + \frac{A_4}{2}$$

$$a_4'' = -\frac{1}{2} \bar{A}_0$$

$$a_5' = \frac{A_1}{2} - A_3 + \frac{A_5}{2}$$

$$a_5'' = -\frac{1}{2} \bar{A}_1$$

$$a_6' = \frac{A_6}{2}$$

$$a_6'' = -\frac{1}{2} \bar{A}_2$$

$$a_7' = \frac{A_0}{6} - \frac{A_2}{2} + \frac{A_4}{2} - \frac{A_7}{6}$$

$$a_7'' = -\frac{1}{2} (\bar{A}_0 - \bar{A}_3)$$

$$a_8' = \frac{A_1}{6} - \frac{A_3}{2} + \frac{A_5}{2} - \frac{A_8}{6}$$

$$a_8'' = -\frac{1}{2} (\bar{A}_1 - \bar{A}_4)$$

$$a_9' = \frac{A_6}{2} - \frac{A_9}{6}$$

$$a_9'' = -\frac{1}{2} (\bar{A}_2 - \bar{A}_5)$$



$$b_0' = B_0$$

$$b_1' = B_1$$

$$b_2' = B_0 - B_2$$

$$b_3' = B_1 - B_3$$

$$b_4' = \frac{B_0}{2} - B_2 + \frac{B_4}{2}$$

$$b_5' = \frac{B_1}{2} - B_3 + \frac{B_5}{2}$$

$$b_6' = \frac{B_6}{2}$$

$$b_7' = \frac{B_0}{6} - \frac{B_2}{2} + \frac{B_4}{2} - \frac{B_7}{6}$$

$$b_8' = \frac{B_1}{6} - \frac{B_3}{2} + \frac{B_5}{2} - \frac{B_8}{6}$$

$$b_9' = \frac{B_6}{2} - \frac{B_9}{6}$$

$$b_0'' = 0$$

$$b_1'' = 0$$

$$b_2'' = 0$$

$$b_3'' = 0$$

$$b_4'' = -\frac{1}{2} \bar{B}_0$$

$$b_5'' = -\frac{1}{2} \bar{B}_1$$

$$b_6'' = -\frac{1}{2} \bar{B}_2$$

$$b_7'' = -\frac{1}{2} (\bar{B}_0 - \bar{B}_3)$$

$$b_8'' = -\frac{1}{2} (\bar{B}_1 - \bar{B}_4)$$

$$b_9'' = -\frac{1}{2} (\bar{B}_2 - \bar{B}_5)$$

$$c_0' = C_0$$

$$c_1' = C_0 - C_1$$

$$c_2' = -C_2$$

$$c_3' = \frac{C_0}{2} - C_1 + \frac{C_3}{2}$$

$$c_4' = -C_2 + \frac{C_4}{2}$$

$$c_5' = \frac{C_0}{6} - \frac{C_1}{2} + \frac{C_3}{2} - \frac{C_5}{6}$$

$$c_6' = -\frac{C_2}{2} + \frac{C_4}{2} - \frac{C_6}{6}$$

$$c_7' = -\frac{C_7}{6}$$

$$c_0'' = 0$$

$$c_1'' = 0$$

$$c_2'' = 0$$

$$c_3'' = -\frac{1}{2} \bar{C}_0$$

$$c_4'' = -\frac{1}{2} \bar{C}_1$$

$$c_5'' = -\frac{1}{2} (\bar{C}_0 - \bar{C}_2)$$

$$c_6'' = -\frac{1}{2} (\bar{C}_1 - \bar{C}_3)$$

$$c_7'' = \frac{1}{2} \bar{C}_4$$

$$d_0' = D_0$$

$$d_1' = D_0 - D_1$$

$$d_2' = -D_2$$

$$d_3' = \frac{D_0}{2} - D_1 + \frac{D_3}{2}$$

$$d_4' = -D_2 + \frac{D_4}{2}$$

$$d_5' = \frac{D_0}{6} - \frac{D_1}{2} + \frac{D_3}{2} - \frac{D_5}{6}$$

$$d_6' = -\frac{D_2}{2} + \frac{D_4}{2} - \frac{D_6}{6}$$

$$d_7' = -\frac{D_7}{6}$$

$$d_0'' = 0$$

$$d_1'' = 0$$

$$d_2'' = 0$$

$$d_3'' = -\frac{1}{2} \bar{D}_0$$

$$d_4'' = -\frac{1}{2} \bar{D}_1$$

$$d_5'' = -\frac{1}{2} (\bar{D}_0 - \bar{D}_2)$$

$$d_6'' = -\frac{1}{2} (\bar{D}_1 - \bar{D}_3)$$

$$d_7'' = \frac{1}{2} \bar{D}_4$$

$$e_0' = E_0$$

$$e_1' = E_0 - E_1$$

$$e_2' = \frac{E_0}{2} - E_1 + \frac{E_2}{2}$$

$$e_3' = \frac{E_3}{2}$$

$$e_4' = \frac{E_0}{6} - \frac{E_1}{2} + \frac{E_2}{2} - \frac{E_4}{6}$$

$$e_5' = \frac{E_3}{2} - \frac{E_5}{6}$$

$$e_0'' = 0$$

$$e_1'' = 0$$

$$e_2'' = -\frac{1}{2} \bar{E}_0$$

$$e_3'' = -\frac{1}{2} \bar{E}_1$$

$$e_4'' = -\frac{1}{2} (\bar{E}_0 - \bar{E}_2)$$

$$e_5'' = -\frac{1}{2} (\bar{E}_1 - \bar{E}_3)$$

$$f_0' = F_0$$

$$f_0'' = 0$$

$$f_1' = F_0 - F_1$$

$$f_1'' = 0$$

$$f_2' = \frac{F_0}{2} - F_1 + \frac{F_2}{2}$$

$$f_2'' = -\frac{1}{2}\bar{F}_0$$

$$f_3' = \frac{F_3}{2}$$

$$f_3'' = -\frac{1}{2}\bar{F}_1$$

$$f_4' = \frac{F_0}{6} - \frac{F_1}{2} + \frac{F_2}{2} - \frac{F_4}{6}$$

$$f_4'' = -\frac{1}{2}(\bar{F}_0 - \bar{F}_2)$$

$$f_5' = \frac{F_3}{2} - \frac{F_5}{6}$$

$$f_5'' = -\frac{1}{2}(\bar{F}_1 - \bar{F}_3)$$

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TABLE I.- COEFFICIENTS IN THE EXPRESSIONS FOR THE VELOCITY POTENTIAL

$a_0$	$a_0'$	$a_1'$	$a_2'$	$a_3'$	$a_4'$	$a_5'$	$a_6'$	$a_7'$	$a_8'$	$a_9'$	$a_4''$	$a_5''$	$a_6''$	$a_7''$	$a_8''$	$a_9''$
0	1.0000	0.1667	0	-0.3333	0	0.1667	0.037500	0	0	-0.02500	0	-0.08333	-0.01250	0	0	0.025000
.03493	.9869	.1636	.0073	-.3220	-.0005	.0759	.035851	-.000017	.001292	-.02302	.000878	-.08052	-.01215	.000071	-.00152	.023709
.08716	.9413	.1532	.0287	-.2864	-.0009	.0538	.030948	-.000137	.003730	-.01773	.003347	-.07178	-.01102	.000489	-.00527	.019971
.17365	.8417	.1311	.0623	-.2201	.0017	.0261	.022539	-.000203	.004732	-.01030	.006913	-.05372	-.00887	.001629	-.00943	.013823
.25882	.7431	.1100	.0821	-.1637	.0056	.0091	.016209	.000132	.003822	-.00583	.008794	-.04253	-.00702	.002614	-.01064	.009396
.34202	.6576	.0923	.0908	-.1238	.0087	.0003	.011848	.000572	.002640	-.00335	.009500	-.03276	-.00558	.003216	-.01033	.006465
.42262	.5868	.0781	.0931	-.0972	.0107	-.0041	.008867	.000942	.001677	-.00192	.009586	-.02362	-.00449	.003320	-.00949	.004336
.50000	.5291	.0669	.0919	-.0768	.0119	-.0060	.006818	.001218	.000956	-.00111	.009379	-.02042	-.00368	.003637	-.00848	.003262
.57358	.4824	.0580	.0894	-.0618	.0125	-.0069	.005366	.001402	.000455	-.00062	.009055	-.01658	-.00307	.003650	-.00733	.002393
.64279	.4443	.0510	.0862	-.0507	.0127	-.0072	.004300	.001520	.000092	-.00026	.008702	-.01369	-.00259	.003606	-.00671	.001776
.70711	.4135	.0453	.0831	-.0424	.0127	-.0072	.003518	.001592	-.000137	-.00011	.008337	-.01153	-.00224	.003534	-.00670	.001336
.76604	.3884	.0406	.0800	-.0364	.0126	-.0067	.002912	.001622	-.000281	-.00003	.008035	-.00991	-.00196	.003460	-.00533	.001083
.81915	.3680	.0376	.0773	-.0316	.0125	-.0064	.002597	.001643	-.000388	.00004	.007733	-.00870	-.00175	.003382	-.00486	.000868
.86605	.3515	.0348	.0750	-.0281	.0123	-.0061	.002287	.001649	-.000434	.00008	.007313	-.00776	-.00159	.003310	-.00446	.000716
.90631	.3385	.0323	.0730	-.0254	.0121	-.0059	.002055	.001650	-.000498	.00011	.007312	-.00704	-.00146	.003247	-.00415	.000604
1.00000	.3118	.0283	.0687	-.0202	.0117	-.0053	.001617	.001638	-.000559	.00015	.006872	-.00566	-.00121	.003099	-.00350	.000404

$a_0$	$b_0'$	$b_1'$	$b_2'$	$b_3'$	$b_4'$	$b_5'$	$b_6'$	$b_7'$	$b_8'$	$b_9'$	$b_4''$	$b_5''$	$b_6''$	$b_7''$	$b_8''$	$b_9''$
0	0	0.3333	0	0	0	0	0.083333	0	0	0	0	0	-0.008333	0	0	0
.03493	.00020	.3333	.000002	.000005	-.000002	.000005	.008333	.00000001	.000003	.0000007	.0000003	-.000004	-.008333	.00000002	-.0000004	-.0000006
.08716	.00116	.3333	.000051	.000031	-.0000014	-.000017	.008322	.00000028	.000005	.0000118	.0000060	-.000012	-.008330	.00000091	-.0000007	-.0000081
.17365	.00395	.3331	.000424	.000720	.0000136	-.000225	.008241	.00000148	.000020	.0000868	.0000478	.000051	-.008310	.00001180	-.0000549	-.0000686
.25882	.00737	.3325	.001212	.002260	.0000933	-.000628	.008058	.00000225	.000017	.0002253	.0001324	.000271	-.008259	.00004165	-.0001163	-.0001976
.34202	.01080	.3316	.002291	.004641	.0002515	-.001050	.007792	.00001692	-.000023	.0003940	.0002446	.000645	-.008178	.00008898	-.0001535	-.0003779
.42262	.01395	.3302	.003508	.007641	.0004709	-.001494	.007474	.00004278	-.000100	.0005654	.0003689	.001130	-.008071	.00014796	-.0001432	-.0005862
.50000	.01672	.3285	.004750	.011010	.0007299	-.001760	.007132	.00007225	-.000198	.0007178	.0004940	.001678	-.007947	.00021291	-.0000789	-.0008008
.57358	.01909	.3266	.005950	.014541	.0010074	-.001869	.006788	.00012041	-.000291	.0008466	.0006136	.002250	-.007812	.00027939	.0000305	-.0010185
.64279	.02108	.3246	.007073	.018060	.0012842	-.001859	.006451	.00016775	-.000372	.0009533	.0007246	.002818	-.007675	.00034581	.0001667	-.0012075
.70711	.02273	.3225	.008073	.021372	.0015594	-.001662	.006149	.00021407	-.000435	.0010343	.0008228	.003346	-.007541	.00040529	.0003399	-.0013743
.76604	.02411	.3205	.008964	.024451	.0018142	-.001407	.005873	.00026100	-.000479	.0010942	.0009098	.003834	-.007411	.00046130	.0005159	-.0015236
.81915	.02523	.3185	.009737	.027237	.0020460	-.001100	.005628	.00030573	-.000502	.0011386	.0009845	.004265	-.007292	.00051145	.0006896	-.0016550
.86605	.02613	.3167	.010393	.029665	.0022510	-.000772	.005418	.00034670	-.000511	.0011697	.0010476	.004639	-.007183	.00055453	.0008487	-.0017559
.90631	.02684	.3151	.010937	.031724	.0024265	-.000453	.005242	.00038291	-.000509	.0011914	.0010997	.004955	-.007091	.00059120	.0009994	-.0018414
1.00000	.02829	.3112	.012126	.036378	.0028294	.000405	.004650	.0004652	-.000466	.0012247	.0012126	.005659	-.006872	.00067360	.0013475	-.0020211

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TABLE I.- COEFFICIENTS IN THE EXPRESSIONS FOR THE VELOCITY POTENTIAL - Continued

$\rho_0$	$c_0'$	$c_1'$	$c_2'$	$c_3'$	$c_4'$	$c_5'$	$c_6'$	$c_7'$	$c_3''$	$c_4''$	$c_5''$	$c_6''$	$c_7''$
0	0.5000	0	-0.04167	0	0.020833	0	0	-0.002083	0	-0.020833	0	0	0.002083
.03493	.4995	.00043	-.04135	-.000061	.020665	-.0000003	.000054	-.002068	.000063	-.020777	.0000004	-.00005	.002074
.08716	.4972	.00255	-.04099	-.000332	.019842	-.0000082	.000301	-.001993	.000372	-.020494	.0000099	-.00032	.002028
.17365	.4895	.00909	-.03920	-.000903	.017425	-.0000652	.000903	-.001770	.001300	-.019606	.0000977	-.00105	.001889
.25882	.4782	.01765	-.03675	-.001144	.014463	-.0001772	.001432	-.001523	.002469	-.018398	.0003143	-.00188	.001701
.34202	.4645	.02657	-.03404	-.000846	.011562	-.0002763	.001750	-.001280	.003639	-.017060	.0006401	-.00261	.001326
.42262	.4497	.03483	-.03132	-.000086	.009013	-.0003189	.001869	-.001069	.004680	-.015723	.0010218	-.00317	.001352
.50000	.4347	.04193	-.02877	.000953	.006905	-.0002939	.001852	-.000894	.005544	-.014462	.0014095	-.00356	.001197
.57358	.4200	.04773	-.02646	.002097	.005218	-.0002139	.001756	-.000755	.006229	-.013313	.0017706	-.00381	.001063
.64279	.4062	.05232	-.02442	.003220	.003894	-.0000997	.001625	-.000644	.006755	-.012294	.0020886	-.00395	.000951
.70711	.3934	.05585	-.02263	.004249	.002867	.0000295	.001484	-.000556	.007151	-.011405	.0023583	-.00401	.000856
.76604	.3820	.05851	-.02115	.005150	.002075	.0001599	.001349	-.000487	.007443	-.010642	.0025808	-.00402	.000779
.81915	.3719	.06048	-.01988	.005913	.001469	.0002816	.001225	-.000433	.007655	-.009997	.0027607	-.00400	.000715
.86603	.3632	.06192	-.01883	.006541	.001008	.0003892	.001117	-.000391	.007806	-.009460	.0029035	-.00396	.000664
.90631	.3559	.06294	-.01798	.007045	.000660	.0004810	.001027	-.000358	.007912	-.009022	.0030145	-.00391	.000623
1.00000	.3395	.06467	-.01617	.008084	0	.0006859	.000833	-.000294	.008084	-.008084	.0032336	-.00377	.000539

$\rho_0$	$d_0'$	$d_1'$	$d_2'$	$d_3'$	$d_4'$	$d_5'$	$d_6'$	$d_7'$	$d_3''$	$d_4''$	$d_5''$	$d_6''$	$d_7''$
0	1.0000	0	-0.1667	0	0.08333	0	0	-0.01250	0	-0.08334	0	0	0.01250
.03493	.9929	.0060	-.1636	-.00063	.07921	-.000038	.001105	-.01195	.00096	-.08180	.00009	-.00129	.01215
.08716	.9669	.0255	-.1532	-.00158	.06662	-.000233	.003705	-.01032	.00395	-.07666	.00060	-.00488	.01102
.17365	.9043	.0626	-.1311	.00016	.04448	-.000498	.006119	-.00751	.00917	-.06594	.00226	-.01023	.00887
.25882	.8348	.0917	-.1100	.00481	.02786	-.000250	.006259	-.00540	.01286	-.05574	.00406	-.01318	.00702
.34202	.7680	.1105	-.0923	.00981	.01677	.000372	.005493	-.00395	.01499	-.04709	.00549	-.01433	.00558
.42262	.7078	.1210	-.0781	.01399	.00958	.001092	.004545	-.00296	.01604	-.04005	.00645	-.01442	.00449
.50000	.6552	.1261	-.0669	.01707	.00494	.001732	.003660	-.00227	.01643	-.03438	.00705	-.01396	.00368
.57358	.6101	.1277	-.0580	.01917	.00188	.002238	.002917	-.00179	.01644	-.02984	.00759	-.01326	.00307
.64279	.5717	.1274	-.0510	.02056	-.00017	.002632	.002342	-.00143	.01626	-.02620	.00756	-.01251	.00259
.70711	.5394	.1259	-.0455	.02142	-.00154	.002917	.001855	-.00117	.01597	-.02329	.00761	-.01176	.00224
.76604	.5122	.1239	-.0411	.02193	-.00237	.003108	.001455	-.00100	.01565	-.02097	.00761	-.01107	.00196
.81915	.4897	.1217	-.0376	.02220	-.00299	.003245	.001160	-.00087	.01533	-.01911	.00757	-.01047	.00174
.86603	.4712	.1196	-.0348	.02232	-.00337	.003342	.000926	-.00076	.01503	-.01765	.00751	-.00989	.00159
.90631	.4562	.1178	-.0327	.02235	-.00364	.003406	.000749	-.00069	.01476	-.01649	.00745	-.00945	.00146
1.00000	.4244	.1132	-.0283	.02223	-.00404	.003500	.000404	-.00054	.01415	-.01465	.00728	-.00849	.00121

TABLE I.- COEFFICIENTS IN THE EXPRESSIONS FOR THE VELOCITY POTENTIAL - Concluded

$\rho_0$	$e_0'$	$e_1'$	$e_2'$	$e_3'$	$e_4'$	$e_5'$	$e_2''$	$e_3''$	$e_4''$	$e_5''$
0	0.5000	0	0	0.0208	0	0	0	-0.02083	0	0
.03493	.4998	.00030	-.000062	.0208	-.0000003	.000037	.000063	-.02081	.0000003	-.00004
.08716	.4991	.00184	-.000355	.0205	-.0000074	.000217	.000380	-.02072	.0000087	-.00028
.17365	.4964	.00687	-.001113	.0196	-.0000702	.000725	.001393	-.02040	.0000928	-.00080
.25882	.4923	.01409	-.001781	.0184	-.0002124	.001304	.002794	-.01994	.0003231	-.00154
.34202	.4871	.02251	-.002030	.0170	-.0003946	.001819	.004359	-.01939	.0007201	-.00233
.42262	.4811	.03133	-.001751	.0157	-.0005551	.002216	.005924	-.01879	.0012443	-.00306
.50000	.4746	.03994	-.000995	.0144	-.0006493	.002493	.007389	-.01818	.0018451	-.00371
.57358	.4680	.04796	.000115	.0132	-.0006605	.002670	.008701	-.01758	.0024721	-.00426
.64279	.4614	.05520	.001437	.0122	-.0005942	.002772	.009842	-.01701	.0030863	-.00471
.70711	.4550	.06154	.002845	.0113	-.0004681	.002820	.010813	-.01648	.0036611	-.00507
.76604	.4490	.06699	.004240	.0106	-.0003034	.002833	.011624	-.01599	.0041807	-.00535
.81915	.4435	.07159	.005551	.0099	-.0001205	.002824	.012292	-.01556	.0046372	-.00557
.86603	.4386	.07538	.006732	.0094	.0000635	.002803	.012834	-.01519	.0050276	-.00573
.90631	.4343	.07844	.007792	.0090	.0002357	.002777	.013264	-.01487	.0053521	-.00585
1.00000	.4244	.08488	.010103	.0081	.0006737	.002695	.014147	-.01415	.0060630	-.00606

$\rho_0$	$f_0'$	$f_1'$	$f_2'$	$f_3'$	$f_4'$	$f_5'$	$f_2''$	$f_3''$	$f_4''$	$f_5''$
0	1.0000	0	0	0.08334	0	0	0	-0.08334	0	0
.03493	.9974	.0045	-.00075	.08179	-.000038	.000860	.00107	-.08277	.00011	-.00097
.08716	.9875	.0206	-.00247	.07659	-.000295	.003323	.00473	-.08073	.00078	-.00408
.17365	.9615	.0571	-.00273	.06556	-.000967	.007025	.01237	-.07595	.00320	-.01001
.25882	.9290	.0942	.00123	.05501	-.001192	.009049	.01933	-.07065	.00647	-.01492
.34202	.8941	.1261	.00783	.04615	-.000660	.009792	.02474	-.06549	.00975	-.01839
.42262	.8593	.1514	.01520	.03905	.000402	.009822	.02864	-.06072	.01260	-.02067
.50000	.8257	.1705	.02221	.03344	.001712	.009499	.03133	-.05641	.01490	-.02203
.57358	.7945	.1844	.02836	.02901	.003062	.009043	.03312	-.05259	.01668	-.02274
.64279	.7660	.1943	.03347	.02551	.004302	.008560	.03427	-.04924	.01802	-.02304
.70711	.7407	.2011	.03758	.02273	.005387	.008090	.03497	-.04634	.01900	-.02304
.76604	.7178	.2056	.04085	.02055	.006305	.007638	.03536	-.04384	.01971	-.02288
.81915	.6982	.2085	.04338	.01880	.007060	.007265	.03555	-.04173	.02022	-.02261
.86603	.6814	.2103	.04532	.01742	.007665	.006929	.03560	-.03996	.02058	-.02231
.90631	.6675	.2113	.04678	.01634	.008141	.006658	.03559	-.03851	.02083	-.02201
1.00000	.6366	.2122	.04952	.01415	.009099	.006063	.03537	-.03537	.02122	-.02122



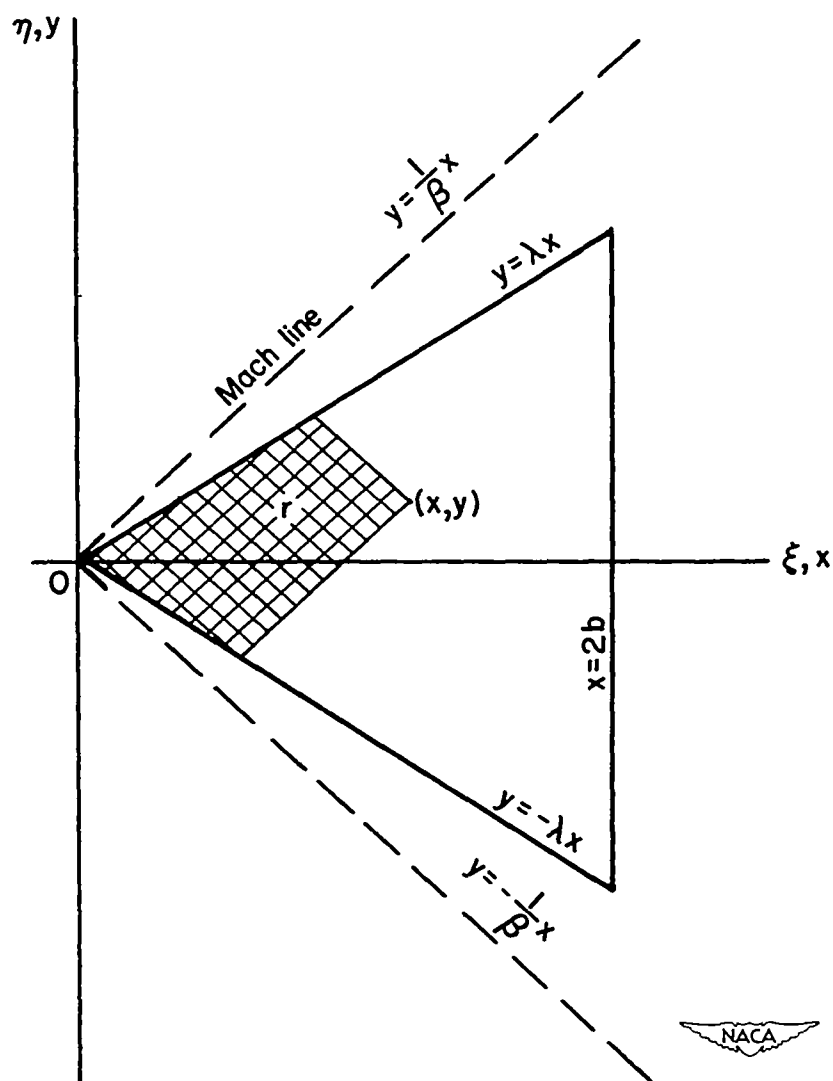
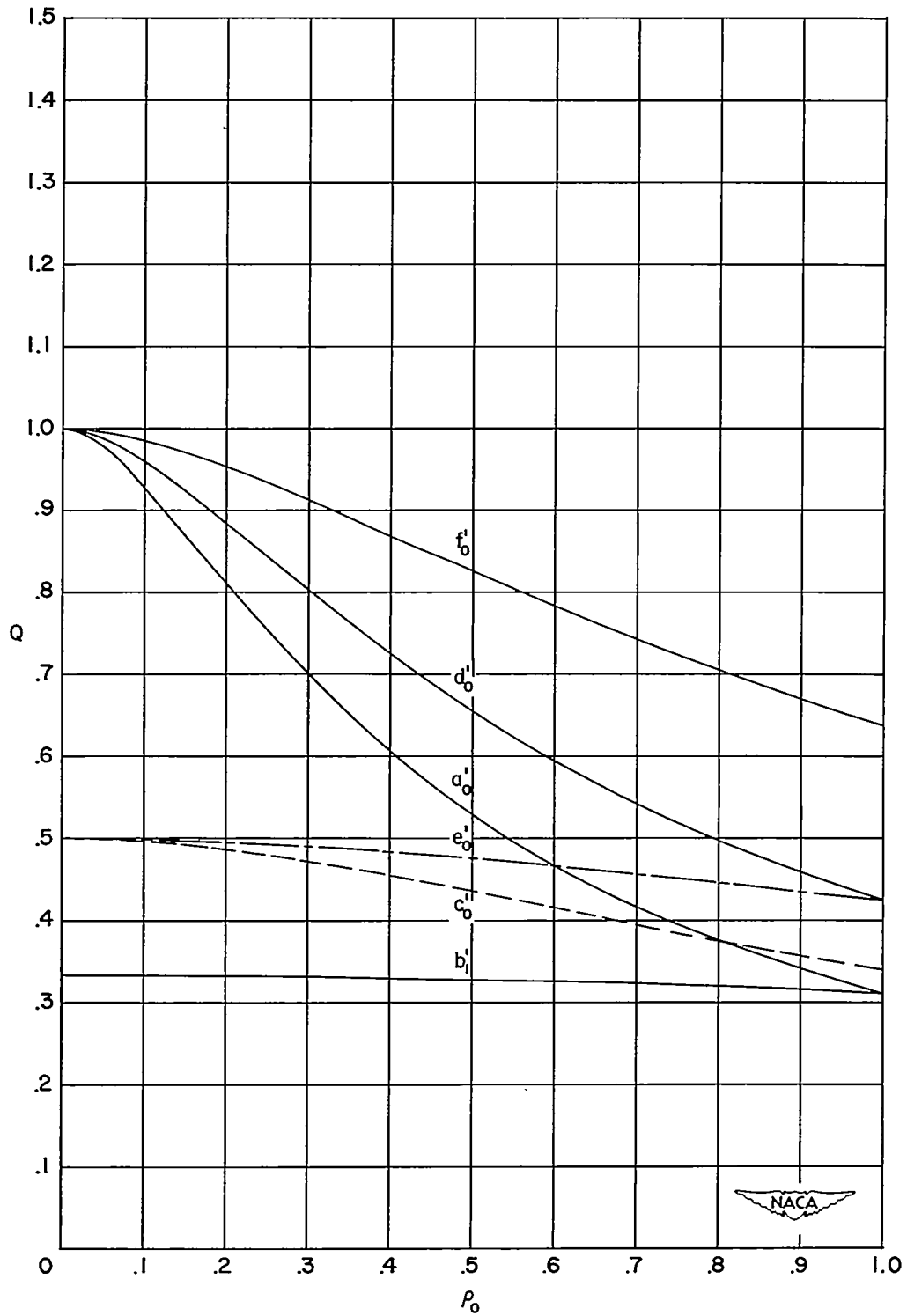


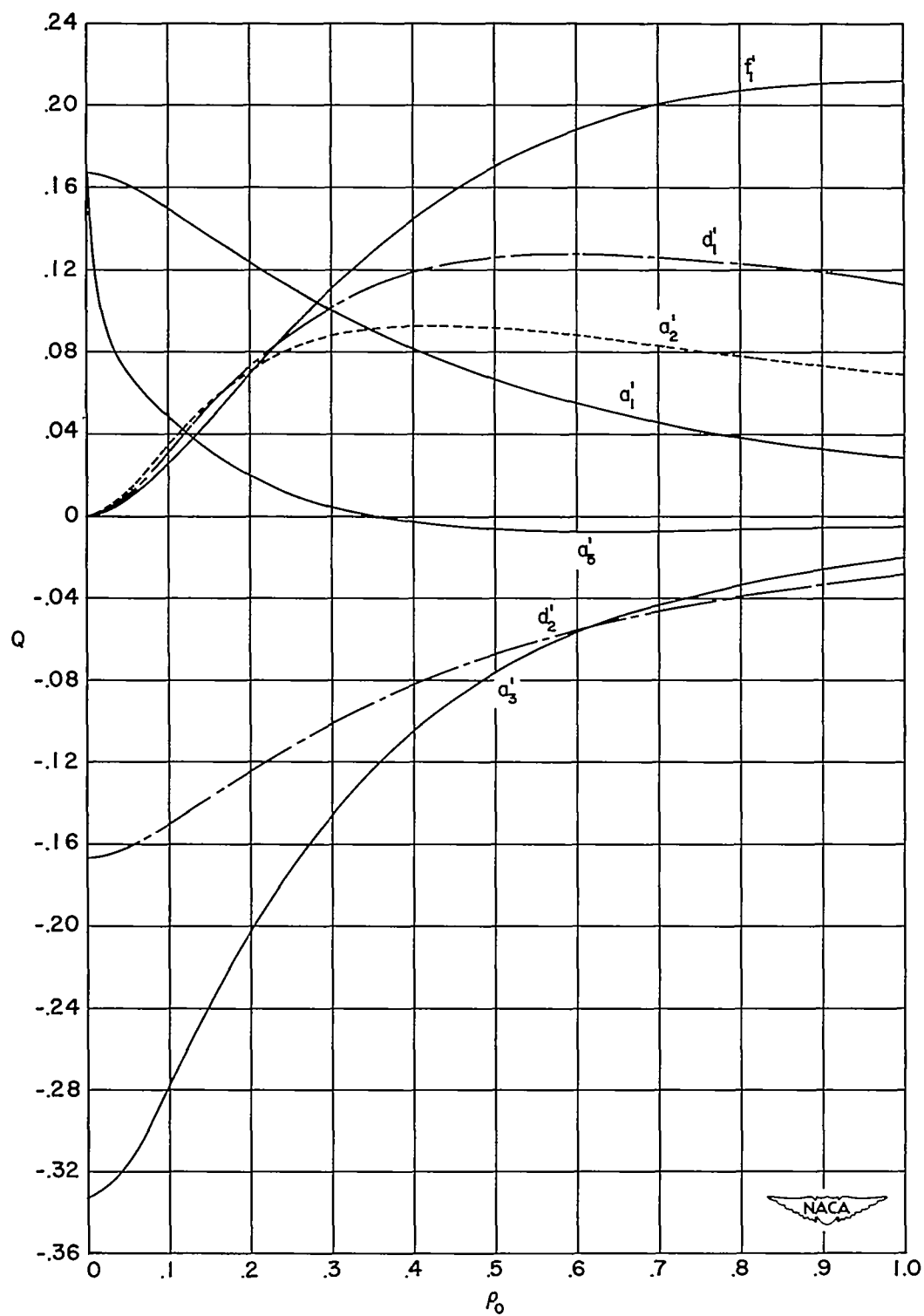
Figure 1.- Sketch illustrating coordinate system.





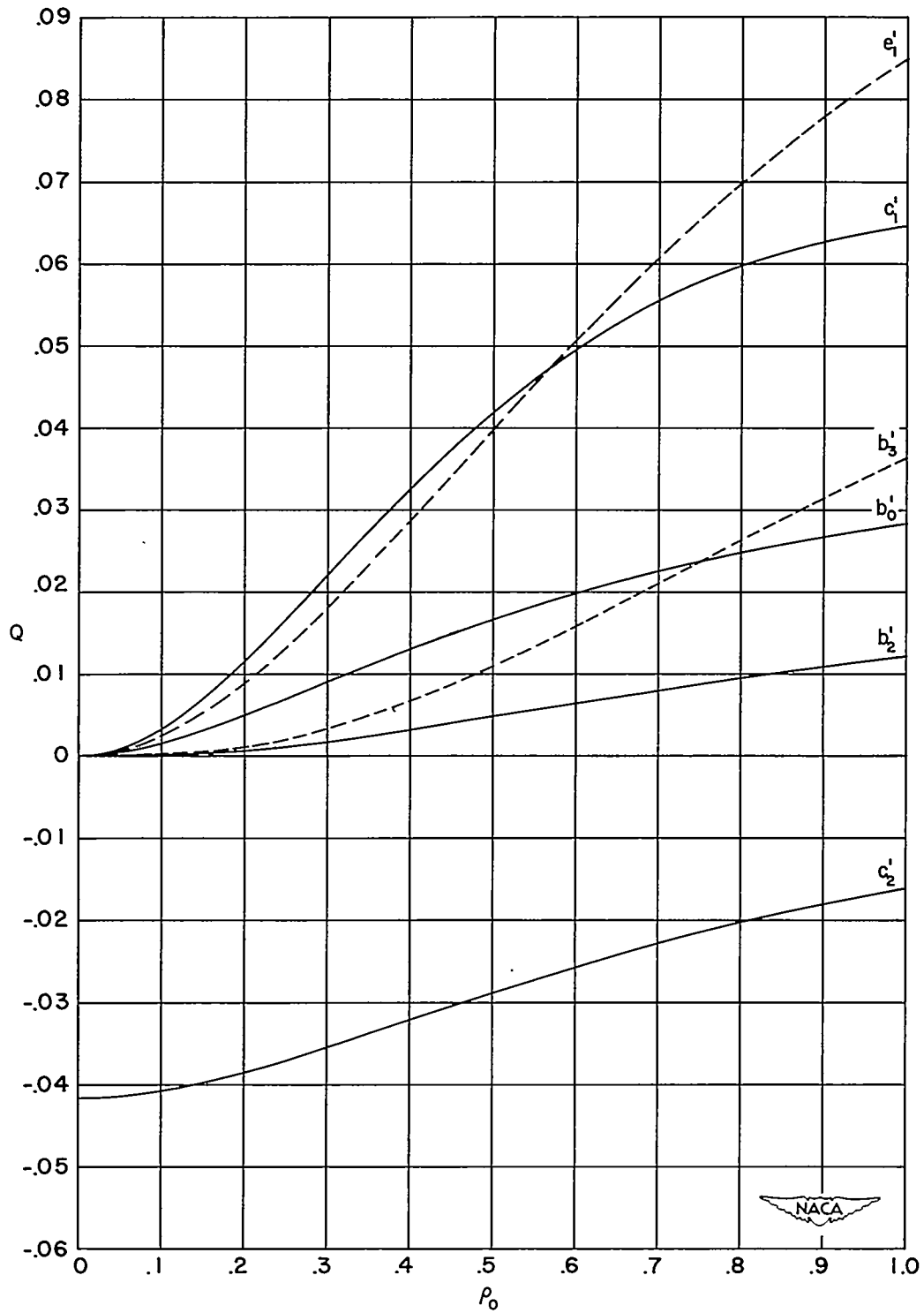
(a)  $Q = a_0', b_1', c_0', d_0', e_0', f_0'$ .

Figure 2.- Variation of the quantities  $Q$  as functions of  $\rho_0$ .



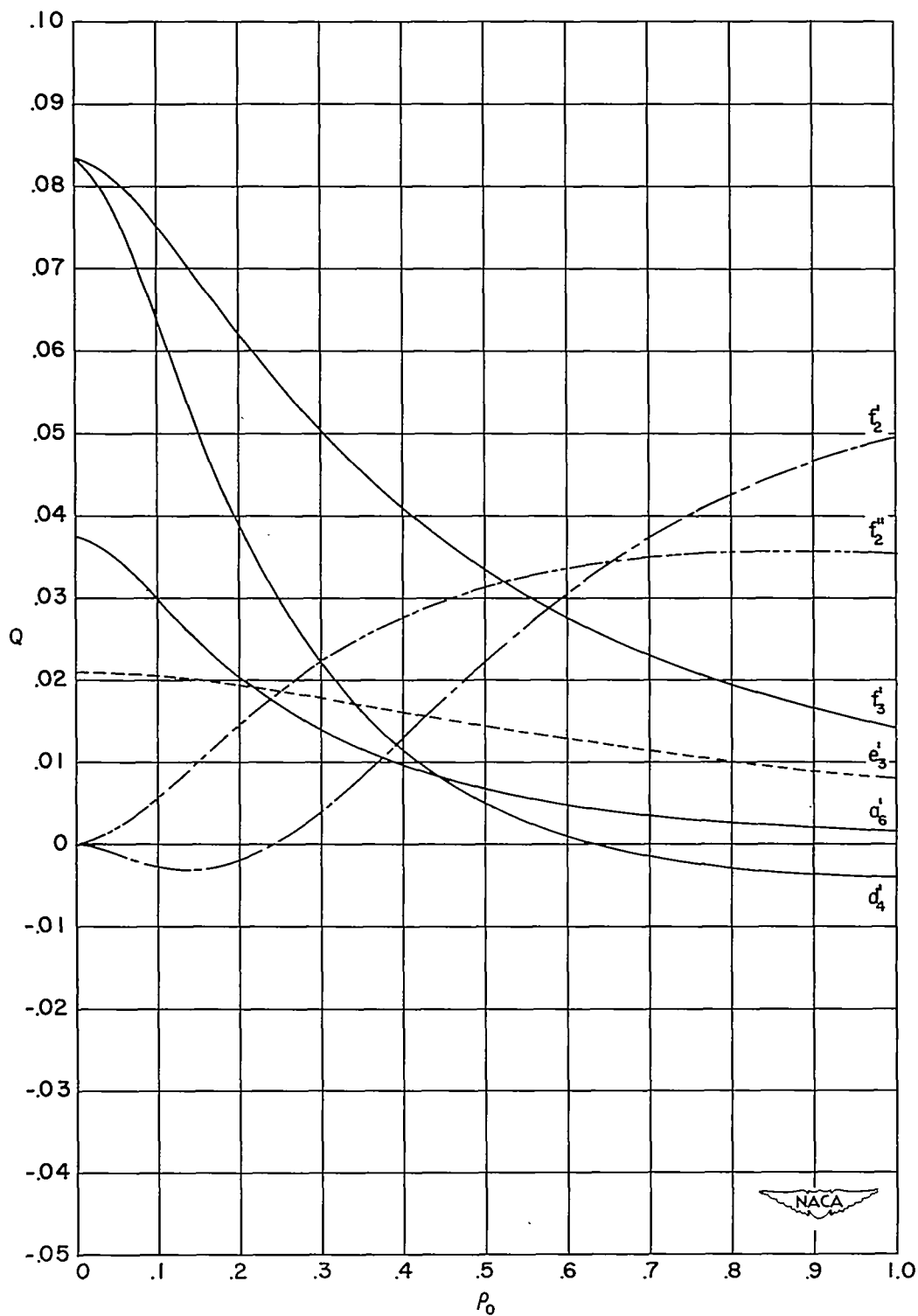
(b)  $Q = a_1', a_2', a_3', a_5', d_1', d_2', f_1'$ .

Figure 2.- Continued.



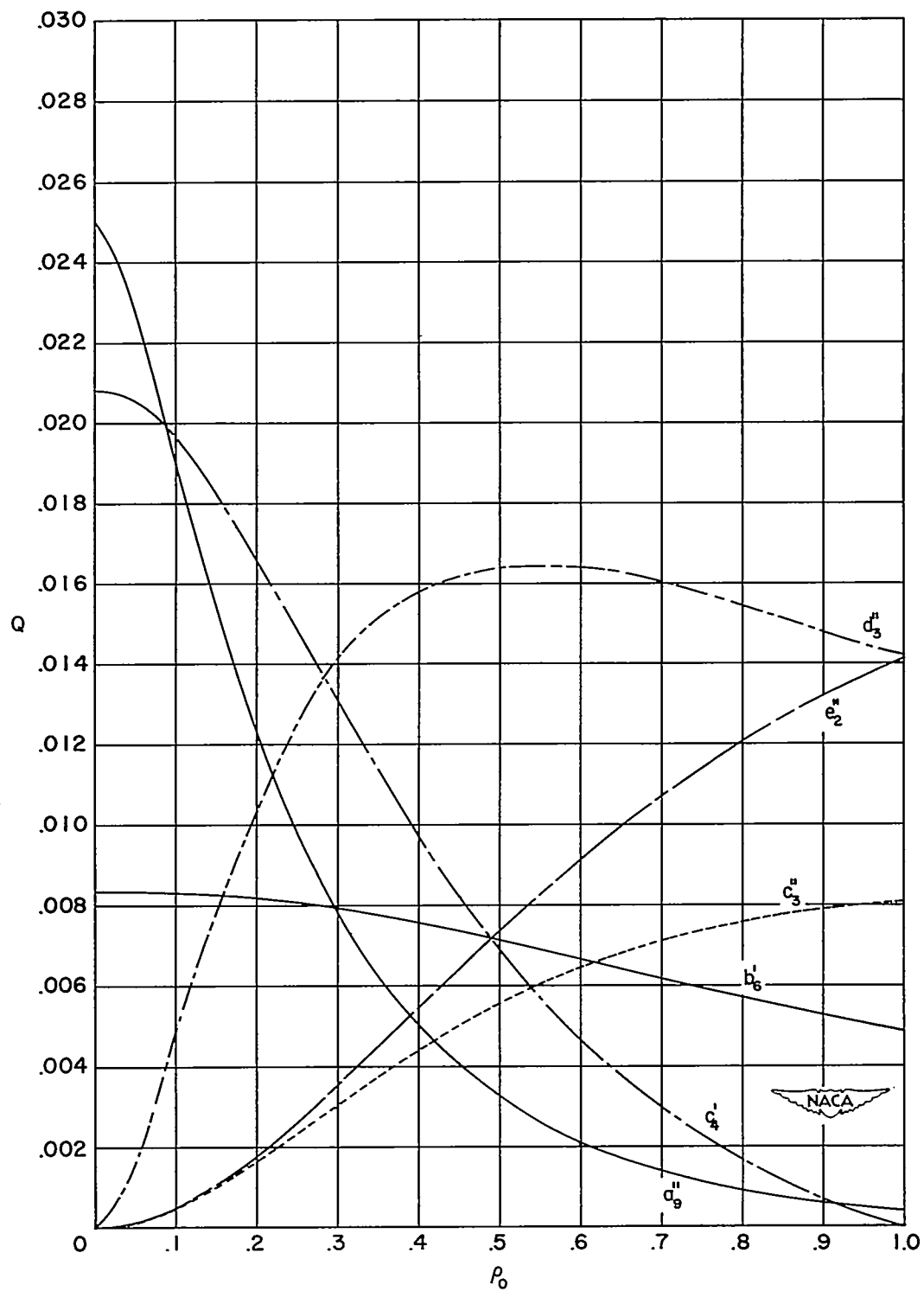
(c)  $Q = b_0', b_2', b_3', c_1', c_2', e_1'$ .

Figure 2.- Continued.



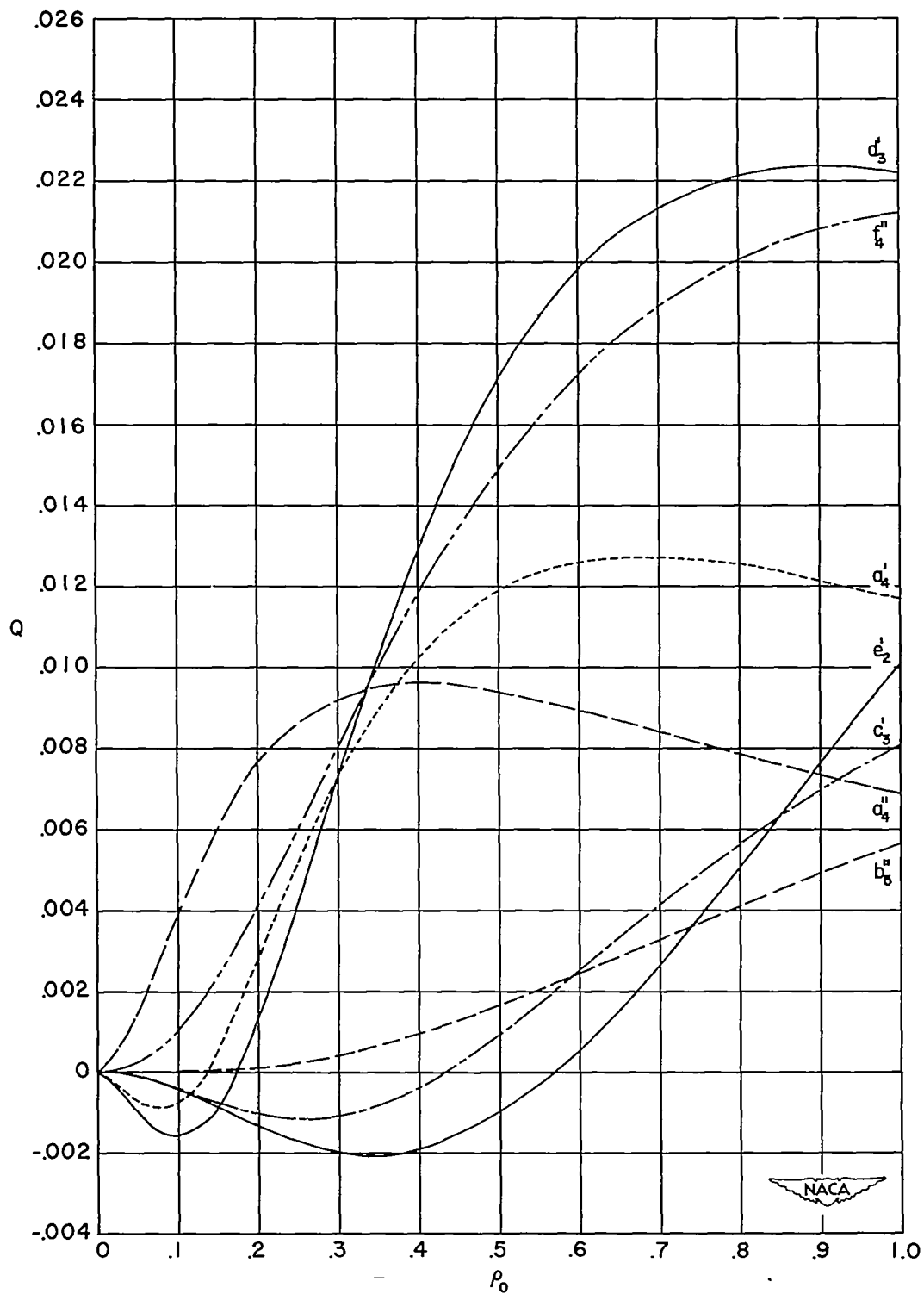
(d)  $Q = a_6', d_4', e_3', f_2', f_3', f_2''$ .

Figure 2.- Continued.



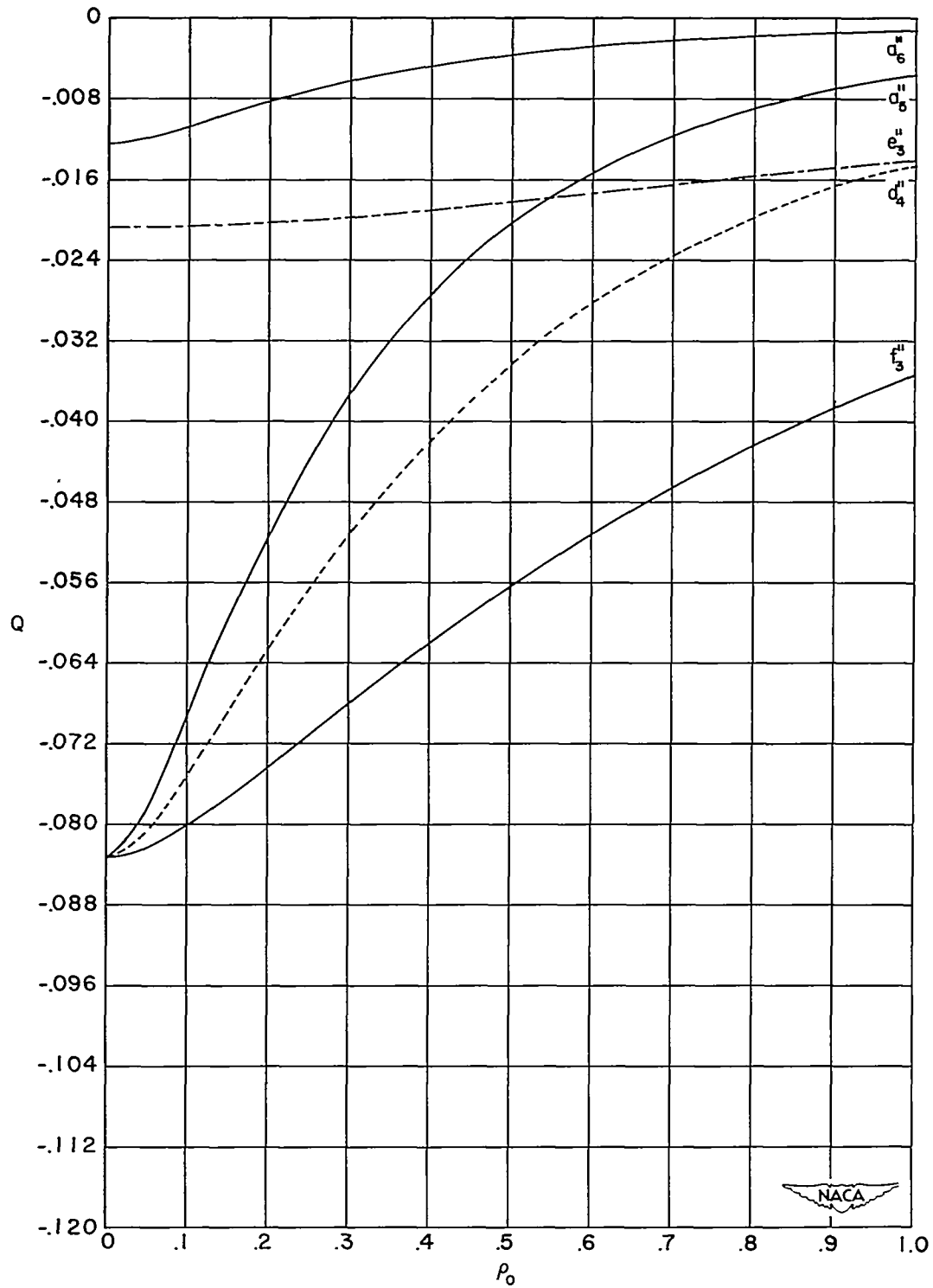
(e)  $Q = b_6', c_4', a_9'', c_3'', d_3'', e_2''$ .

Figure 2.- Continued.



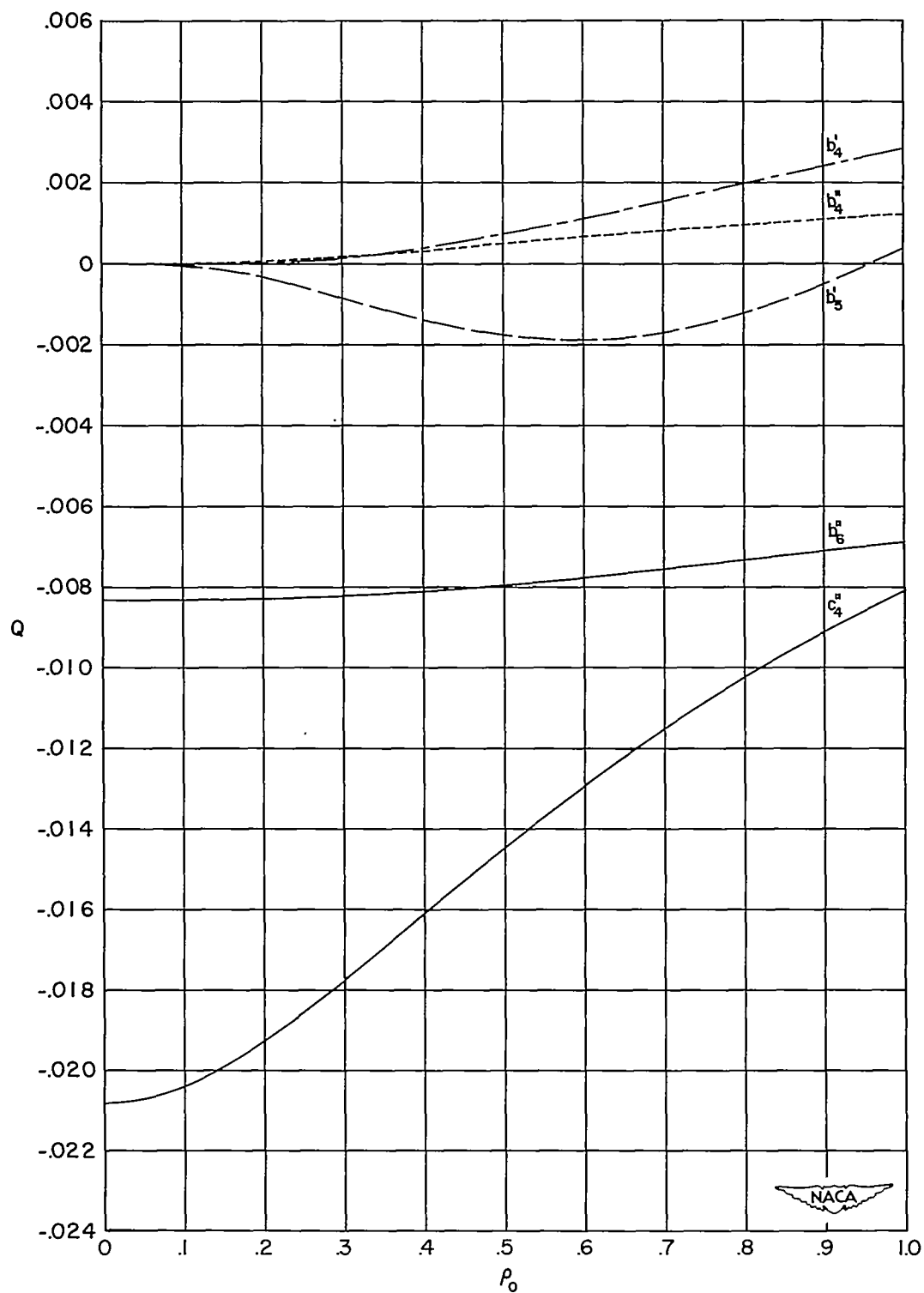
(f)  $Q = a_4', c_3', d_3', e_2', a_4'', b_5'', f_1''$ .

Figure 2.- Continued.



(g)  $Q = a_5'', a_6'', d_4'', e_3'', f_3''$ .

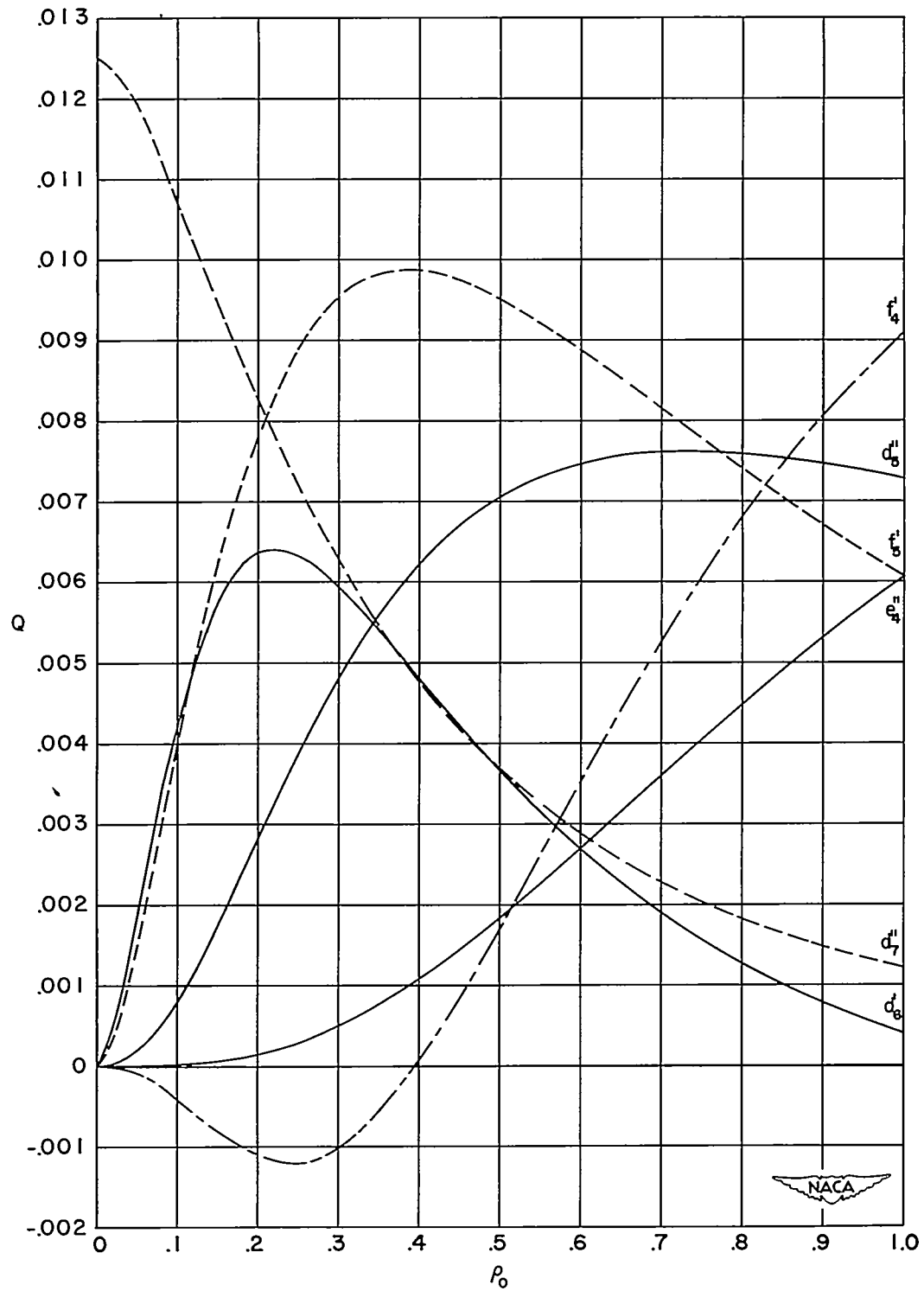
Figure 2.- Continued.



(h)  $Q = b_4', b_5', b_4'', b_6'', c_4''$ .

Figure 2.- Continued.

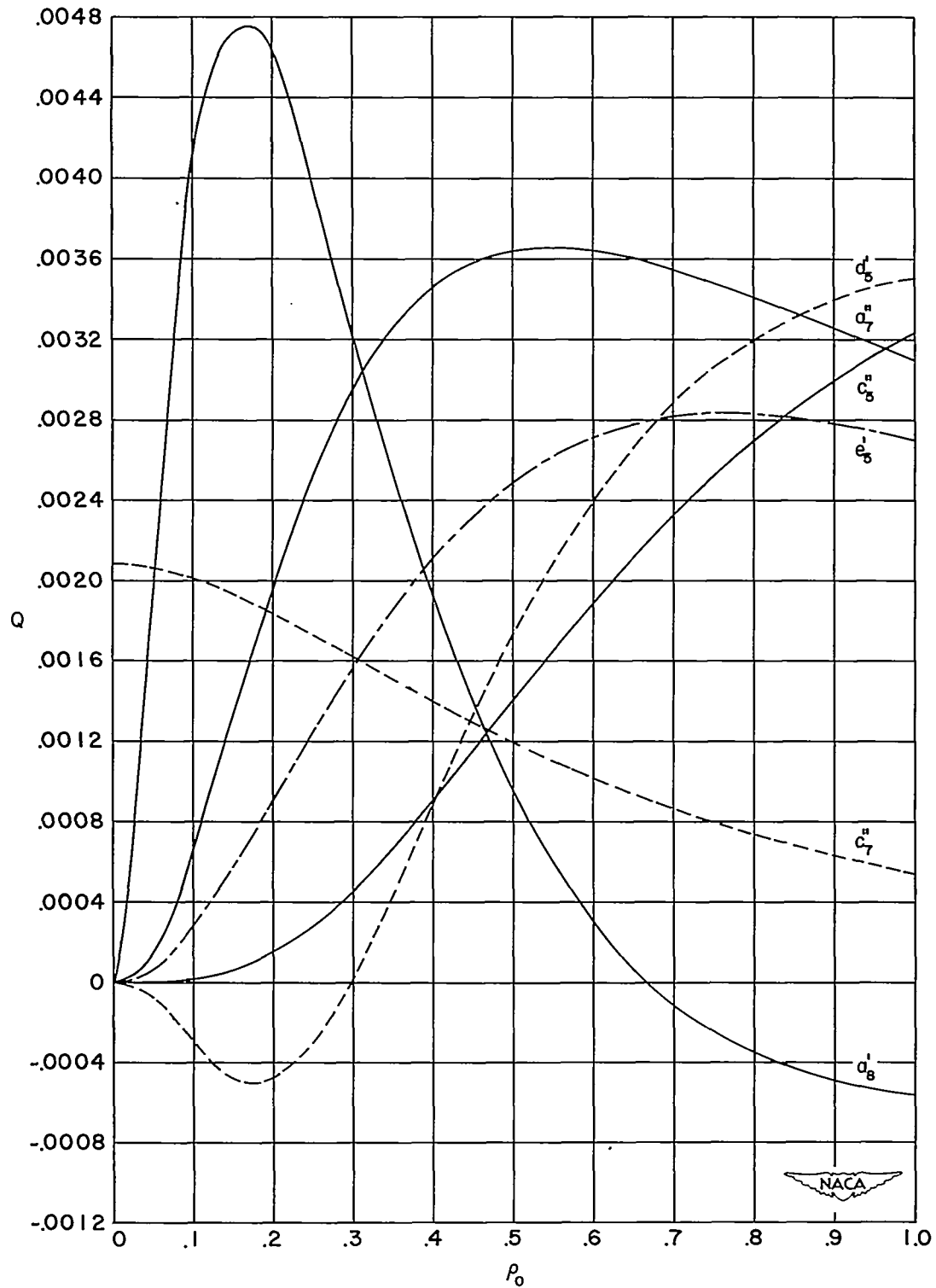




(i)  $Q = d_6', f_4', f_5', d_5'', d_7'', e_4''$ .

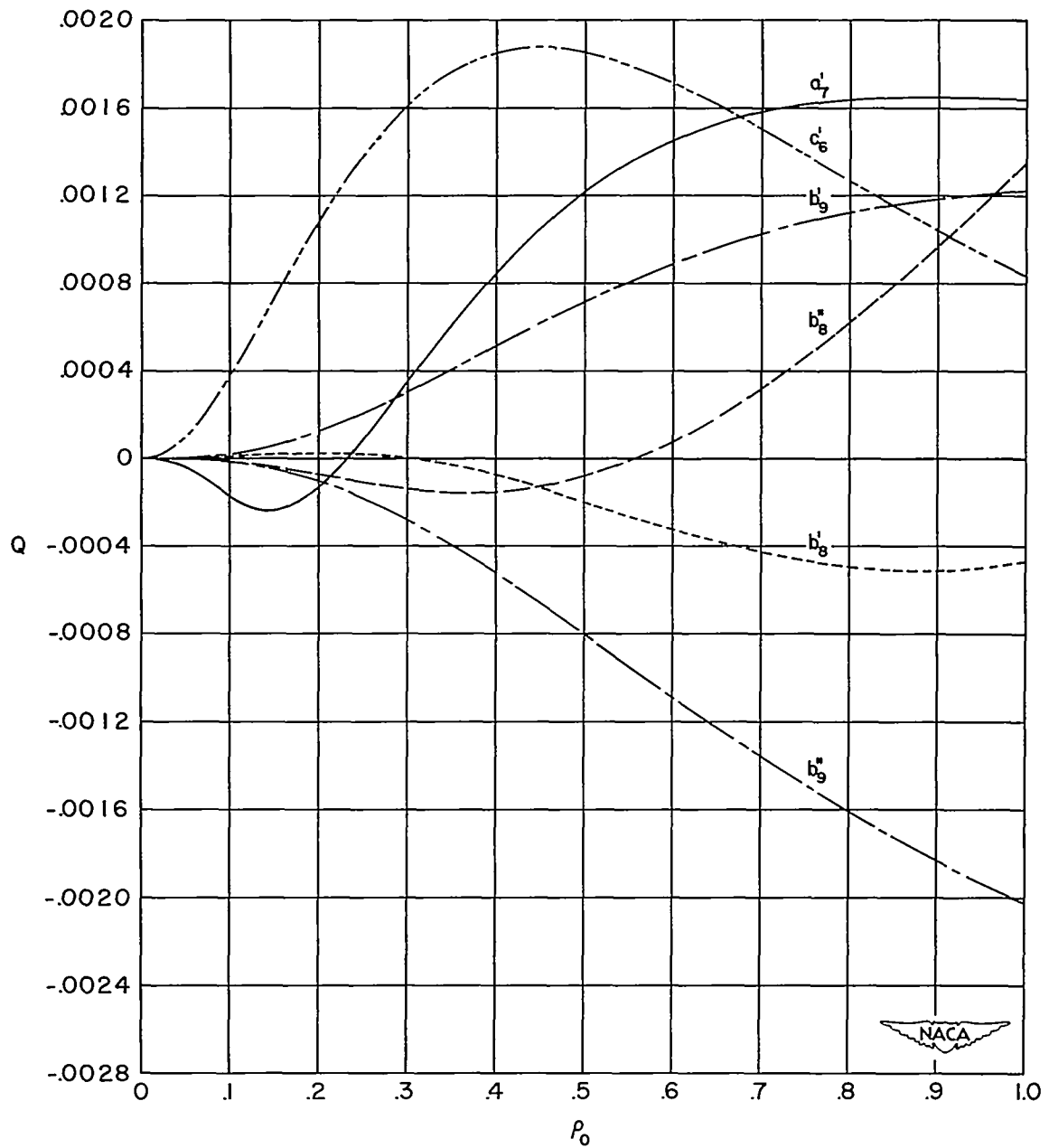
Figure 2.- Continued.

1Z



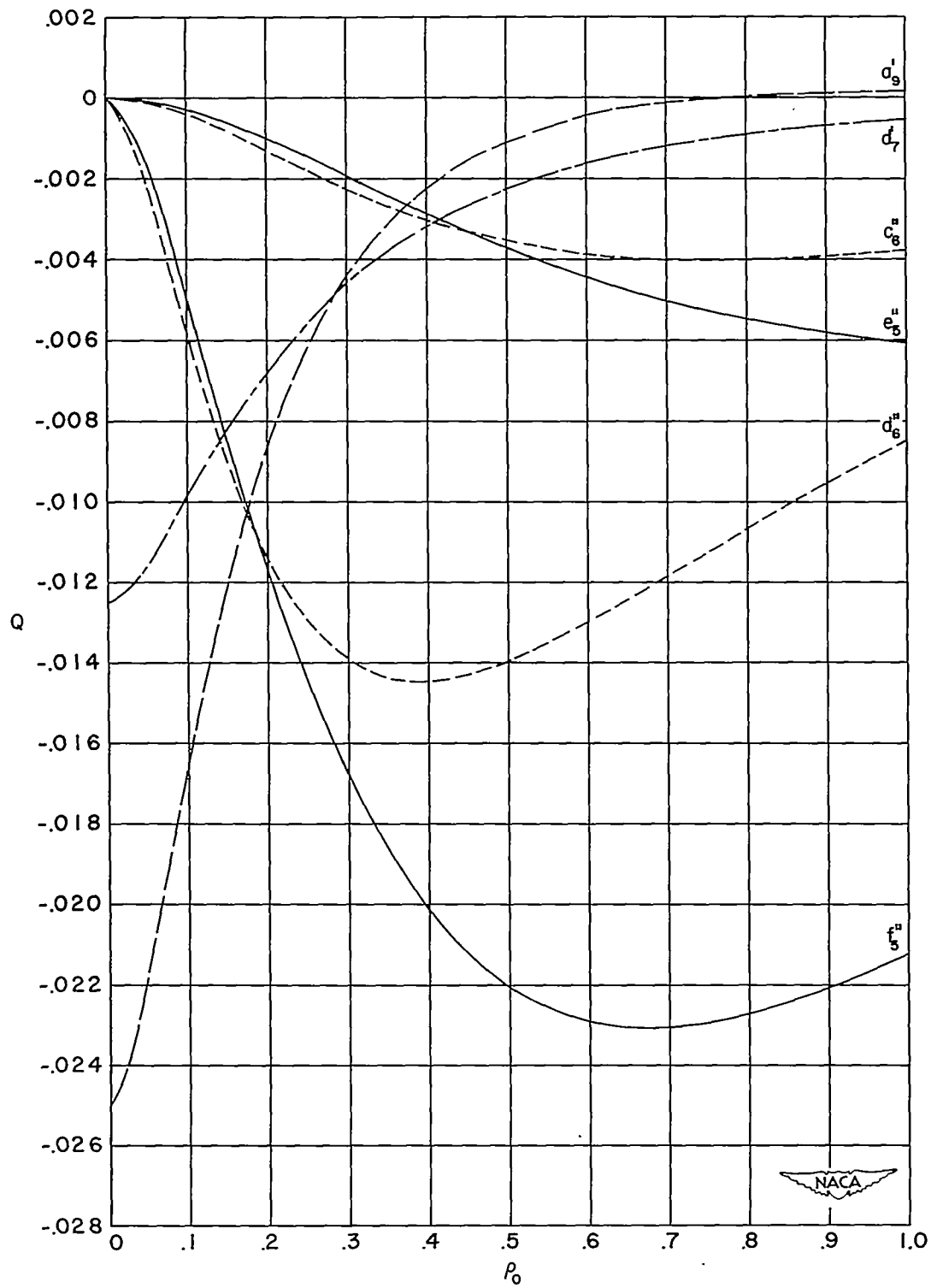
(j)  $Q = a_8', a_5', e_5', a_7'', c_5'', c_7''$ .

Figure 2.- Continued.



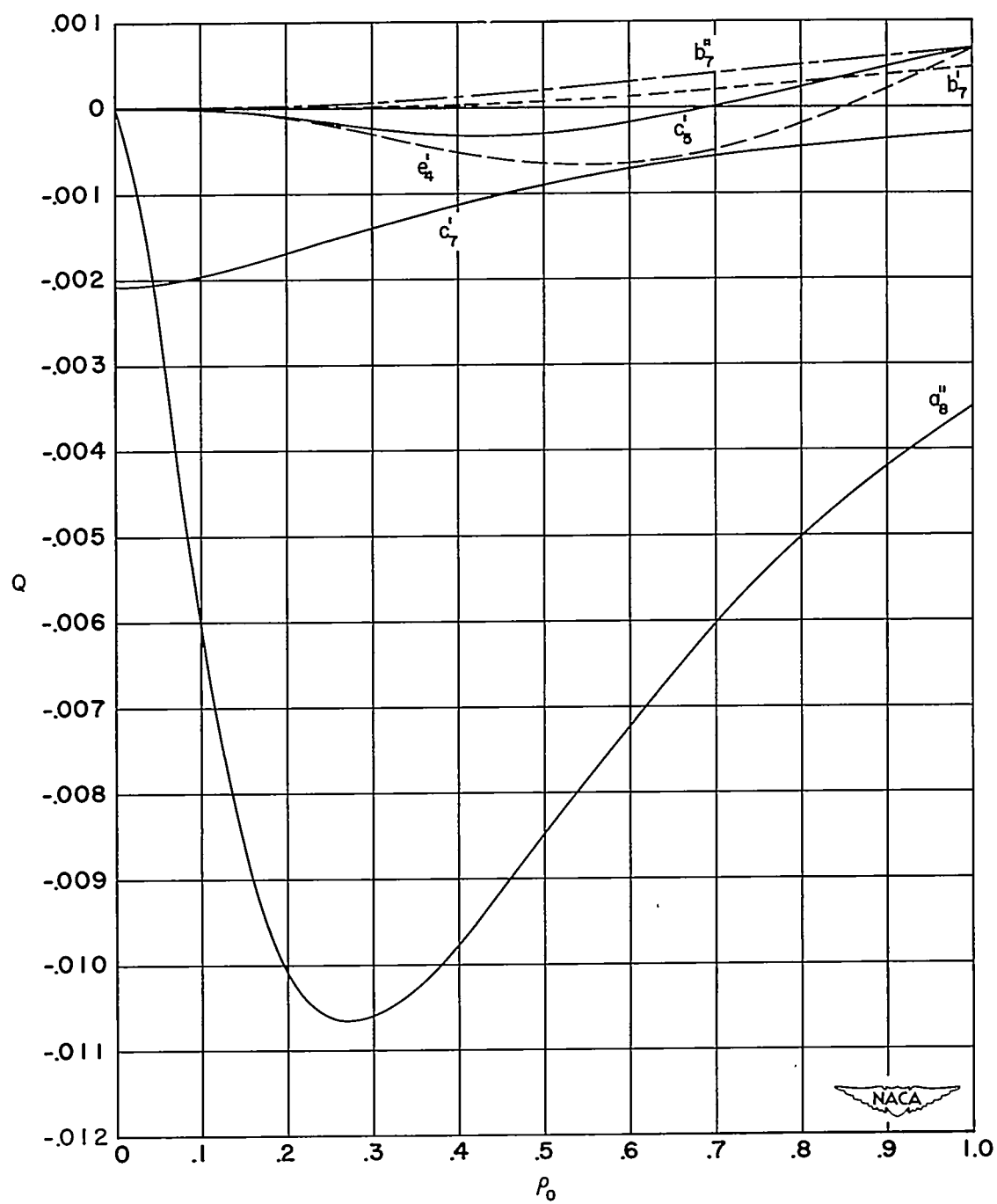
(k)  $Q = a_7', b_8', b_9', c_6', b_8'', b_9''$ .

Figure 2.- Continued.



(1)  $Q = a_9', d_7', c_6'', d_6'', e_5'', f_5''$ .

Figure 2.- Continued.



(m)  $Q = b_7^i, c_5^i, c_7^i, e_4^i, a_8^a, b_7^a$ .

Figure 2.- Concluded.

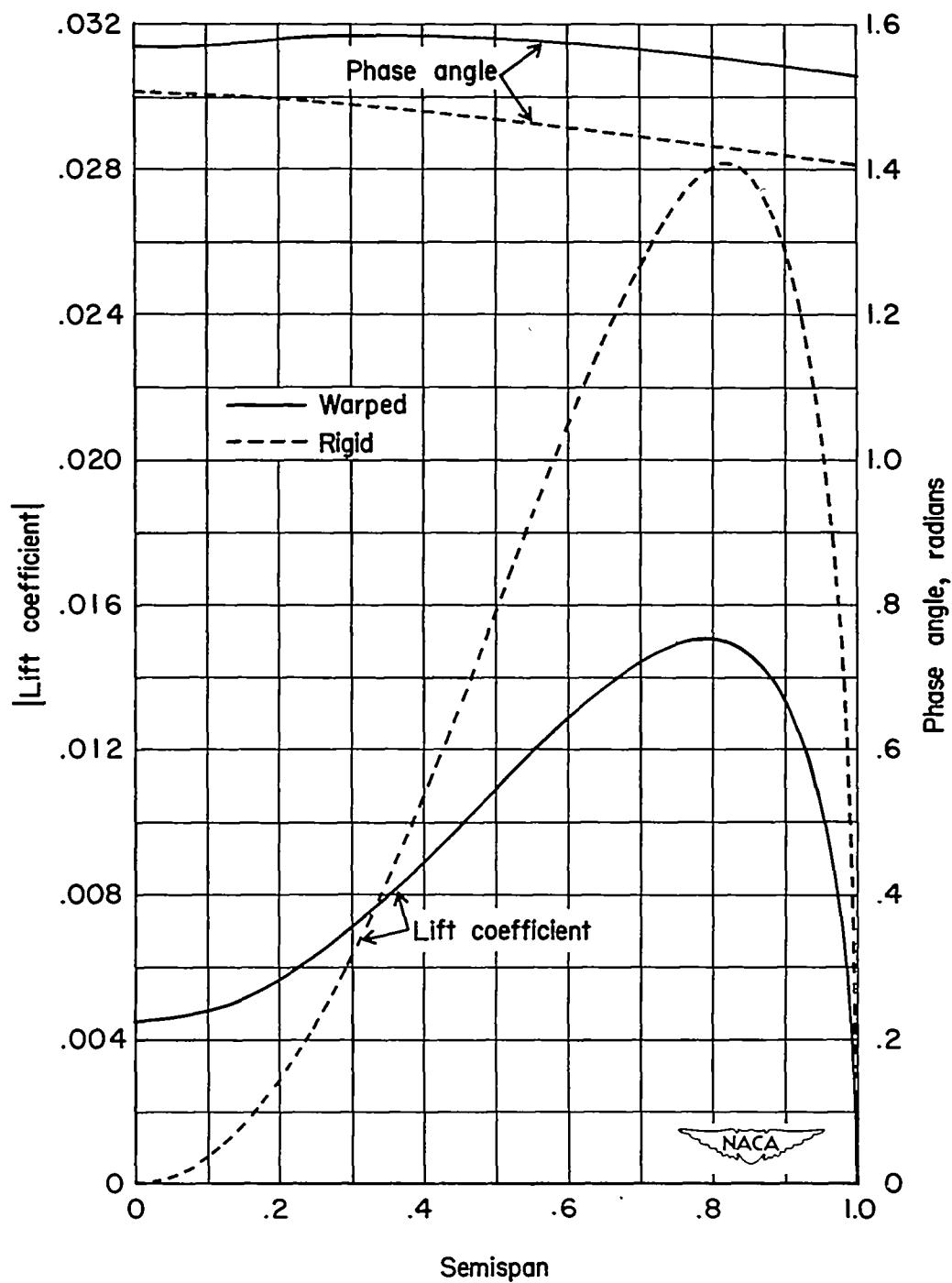


Figure 3.- Phase angles and spanwise variation of lift coefficient associated with parabolic bending compared with the phase angles and the spanwise variation of the product of the absolute value of the lift coefficient, associated with translation, and the ordinates of the parabola  $s = y^2$ .  $45^\circ$  delta wing;  $k = 0.1$ ;  $M = 1.2$ .